

Unbraiding the braided tensor product

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Abstract

We show that the braided tensor product algebra $\mathcal{A}_1 \underline{\otimes} \mathcal{A}_2$ of two module algebras $\mathcal{A}_1, \mathcal{A}_2$ of a quasitriangular Hopf algebra H is equal to the ordinary tensor product algebra of \mathcal{A}_1 with a subalgebra of $\mathcal{A}_1 \underline{\otimes} \mathcal{A}_2$ isomorphic to \mathcal{A}_2 , provided there exists a realization of H within \mathcal{A}_1 . In other words, under this assumption we construct a transformation of generators which ‘decouples’ $\mathcal{A}_1, \mathcal{A}_2$ (i.e. makes them commuting). We apply the theorem to the braided tensor product algebras of two or more quantum group covariant quantum spaces, deformed Heisenberg algebras and q -deformed fuzzy spheres.

1 Introduction and main theorem

As is well known, given two associative unital algebras $\mathcal{A}_1, \mathcal{A}_2$ (over the field \mathbb{C} , say), one can build a new module algebra \mathcal{A} which is as a vector space the tensor product $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ of the two vector spaces (over the same field) by postulating the product law

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2. \quad (1.1)$$

The resulting algebra is the ordinary tensor product algebra. (1.1) is equivalent to the set of relations

$$(a_1 \otimes \mathbf{1}_2)(b_1 \otimes \mathbf{1}_2) = a_1 b_1 \otimes \mathbf{1}_2, \quad (1.2)$$

$$(a_1 \otimes \mathbf{1}_2)(\mathbf{1}_1 \otimes a_2) = a_1 \otimes a_2, \quad (1.3)$$

$$(\mathbf{1}_1 \otimes a_2)(\mathbf{1}_1 \otimes b_2) = \mathbf{1}_1 \otimes a_2 b_2, \quad (1.4)$$

$$(\mathbf{1}_1 \otimes a_2)(a_1 \otimes \mathbf{1}_2) = (a_1 \otimes \mathbf{1}_2)(\mathbf{1}_1 \otimes a_2). \quad (1.5)$$

However, in many cases the same goal can be reached also by replacing (1.5) by some suitable nontrivial commutation relations. With a standard abuse of notation we shall denote in the sequel $a_1 \otimes a_2$ by $a_1 a_2$ for any $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$ and omit all units $\mathbf{1}_i$ when multiplied by non-unit elements; consequently (1.2-1.4) take trivial forms, whereas (1.5) becomes the commutation relation

$$a_2 a_1 = a_1 a_2. \quad (1.6)$$

If $\mathcal{A}_1, \mathcal{A}_2$ are module algebras of a Lie algebra \mathfrak{g} , and we require \mathcal{A} to be too, then (1.6) has no alternative, because any $g \in \mathfrak{g}$ acts as a derivation on the (algebra as well as tensor) product of any two elements or, in Hopf algebra language, because the coproduct $\Delta(g) = g_{(1)} \otimes g_{(2)}$ (at the rhs we have used Sweedler notation) of the Hopf algebra $H \equiv U\mathfrak{g}$ is cocommutative. In the main part of this paper we shall work with right-module algebras (instead of left ones), and denote by $\triangleleft: (a_i, g) \in \mathcal{A}_i \times H \rightarrow a_i \triangleleft g \in \mathcal{A}_i$ the right action; the reason is that they are equivalent to left comodule algebras, which are used in much of the literature. In section 5 we shall give the formulae for left module algebras. We recall that a right action $\triangleleft: (a, g) \in \mathcal{A} \times H \rightarrow a \triangleleft g \in \mathcal{A}$ by definition fulfills

$$a \triangleleft (gg') = (a \triangleleft g) \triangleleft g', \quad (1.7)$$

$$(aa') \triangleleft g = (a \triangleleft g_{(1)}) (a' \triangleleft g_{(2)}). \quad (1.8)$$

If we “ q -deform” this setting by taking as Hopf algebra H the quantum group $U_q \mathfrak{g}$, and as \mathcal{A}_i the corresponding q -deformed module algebras, then it is also known [20, 23, 24] that although $\Delta(g)$ is no longer cocommutative, it is still possible to build the deformed counterpart of \mathcal{A} if one replaces (1.6) with nontrivial commutation relations of the form

$$a_2 a_1 = (a_1 \triangleleft \mathcal{R}^{(1)}) (a_2 \triangleleft \mathcal{R}^{(2)}). \quad (1.9)$$

Here $\mathcal{R} \equiv \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \in H^+ \otimes H^-$ denotes the so-called universal R -matrix of $H \equiv U_q \mathfrak{g}$ [9], and H^\pm denote the Hopf positive and negative Borel subalgebras of H . This yields instead of \mathcal{A} a *braided* tensor product algebra $\mathcal{A}^+ = \mathcal{A}_1 \underline{\otimes}^+ \mathcal{A}_2$ [24]. An alternative one $\mathcal{A}^- = \mathcal{A}_1 \underline{\otimes}^- \mathcal{A}_2$ is obtained by replacing in the previous formula \mathcal{R} by \mathcal{R}_{21}^{-1} :

$$a_2 a_1 = (a_1 \triangleleft \mathcal{R}^{-1(2)}) (a_2 \triangleleft \mathcal{R}^{-1(1)}). \quad (1.10)$$

Both \mathcal{A}^+ and \mathcal{A}^- go to the ordinary tensor product algebra \mathcal{A} in the limit $q \rightarrow 1$.

This is a particular example of a more general notion, that of a *crossed* (or *twisted*) *tensor product* [2] of two unital associative algebras.

In view of (1.9) or (1.10) studying representations of \mathcal{A}^\pm is a more difficult task than just studying the representations of $\mathcal{A}_1, \mathcal{A}_2$, taking their tensor products and studying the irreducible ones there contained. The degrees of freedom of $\mathcal{A}_1, \mathcal{A}_2$ are so to say “coupled”. One might ask whether one can “decouple” them by a transformation of generators.

In this work we present a sufficient condition for the construction of a transformation making \mathcal{A}^+ equal to an ordinary tensor product $\mathcal{A}_1 \otimes \tilde{\mathcal{A}}_2^+$, with $\tilde{\mathcal{A}}_2^+$ a subalgebra of \mathcal{A}^+ isomorphic to \mathcal{A}_2 and *commuting* [in the sense (1.6)] with \mathcal{A}_1 , although - of course - no longer a H -submodule; and similarly for \mathcal{A}^- . In a quantum theory framework one could thus interpret the generators of $\mathcal{A}_1, \tilde{\mathcal{A}}_2^\pm$ as pertaining to decoupled degrees of freedom, describing e.g. some composite or “quasiparticle” excitations. Reducing \mathcal{A}^\pm to a form $\mathcal{A}_1 \otimes \tilde{\mathcal{A}}_2^\pm$ will be called an *unbraiding* of the braided tensor product algebra $\mathcal{A}^\pm = \mathcal{A}_1 \underline{\otimes}^\pm \mathcal{A}_2$. The sufficient condition is that there respectively exists an algebra homomorphism φ_1^+ or an algebra homomorphism φ_1^-

$$\varphi_1^\pm : \mathcal{A}_1 \rtimes H^\pm \rightarrow \mathcal{A}_1 \quad (1.11)$$

acting as the identity on \mathcal{A}_1 , namely for any $a_1 \in \mathcal{A}_1$

$$\varphi_1^\pm(a_1) = a_1. \quad (1.12)$$

(Note that, as a consequence of (1.12), φ_1^\pm is idempotent, $(\varphi_1^\pm)^2 = \varphi_1^\pm$). Here $\mathcal{A}_1 \rtimes H^\pm$ denotes the cross product between \mathcal{A}_1 and H^\pm . In other words, this amounts to assuming that $\varphi_1^+(H^+)$ [resp. $\varphi_1^-(H^-)$] provides a *realization* of H^+ (resp. H^-) within \mathcal{A}_1 . In fact, $\tilde{\mathcal{A}}_2$ is found using the main result of this work:

Theorem 1 *Let $\{H, \mathcal{R}\}$ be a quasitriangular Hopf algebra and H^+, H^- be Hopf subalgebras of H such that $\mathcal{R} \in H^+ \otimes H^-$. Let $\mathcal{A}_1, \mathcal{A}_2$ be respectively a H^+ - and a H^- -module algebra, so that we can define \mathcal{A}^+ as in (1.9), and φ_1^+ be a homomorphism of the type (1.11), (1.12), so that we can define the “unbraiding” map $\chi^+ : \mathcal{A}_2 \rightarrow \mathcal{A}^+$ by*

$$\chi^+(a_2) := \varphi_1^+(\mathcal{R}^{(1)}) (a_2 \triangleleft \mathcal{R}^{(2)}). \quad (1.13)$$

Alternatively, let $\mathcal{A}_1, \mathcal{A}_2$ be respectively a H^- - and a H^+ -module algebra, so that we can define \mathcal{A}^- as in (1.10), and φ_1^- be a homomorphism of the type (1.11), (1.12), so that we can define the “unbraiding” map $\chi^- : \mathcal{A}_2 \rightarrow \mathcal{A}^-$ by

$$\chi^-(a_2) := \varphi_1^-(\mathcal{R}^{-1(2)})(a_2 \triangleleft \mathcal{R}^{-1(1)}). \quad (1.14)$$

In either case χ^\pm are then injective algebra homomorphisms and

$$[\chi^\pm(a_2), \mathcal{A}_1] = 0, \quad (1.15)$$

namely the subalgebras $\tilde{\mathcal{A}}_2^\pm := \chi^\pm(\mathcal{A}_2) \approx \mathcal{A}_2$ commute with \mathcal{A}_1 . Moreover $\mathcal{A}^\pm = \mathcal{A}_1 \otimes \tilde{\mathcal{A}}_2^\pm$.

Proof . We start by recalling the content of the hypotheses stated in the theorem. The algebra $\mathcal{A}_1 \rtimes H^\pm$ as a vector space is the tensor product of \mathcal{A}_1 and H^\pm , whereas its product law is obtained combining the product laws of these two tensor factors with the cross-product law,

$$a_1 g = g_{(1)} (a_1 \triangleleft g_{(2)}), \quad (1.16)$$

for any $a_1 \in \mathcal{A}_1$ and $g \in H^\pm$. φ_1^\pm being an algebra homomorphism means that for any $\xi, \xi' \in \mathcal{A}_1 \rtimes H^\pm$

$$\varphi_1^\pm(\xi \xi') = \varphi_1^\pm(\xi) \varphi_1^\pm(\xi'). \quad (1.17)$$

For $\xi \equiv a \in \mathcal{A}_1 \subset \mathcal{A}_1 \rtimes H^\pm$, $\xi' \equiv g \in H^\pm \subset \mathcal{A}_1 \rtimes H^\pm$ this implies

$$a \varphi^\pm(g) = \varphi^\pm(g_{(1)})(a \triangleleft g_{(2)}) \quad (1.18)$$

Hereby we have also used (1.12) and (1.16). After these preliminaries, note that under the assumption (1.9), for any $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$

$$\begin{aligned} a_1 \chi^+(a_2) &\stackrel{(1.13)}{=} a_1 \varphi_1^+(\mathcal{R}^{(1)})(a_2 \triangleleft \mathcal{R}^{(2)}) \\ &\stackrel{(1.18)}{=} \varphi^+(\mathcal{R}_{(1)}^{(1)})(a_1 \triangleleft \mathcal{R}_{(2)}^{(1)})(a_2 \triangleleft \mathcal{R}^{(2)}) \\ &\stackrel{(A.1.3)}{=} \varphi^+(\mathcal{R}^{(1)})(a_1 \triangleleft \mathcal{R}^{(1')})(a_2 \triangleleft \mathcal{R}^{(2)} \mathcal{R}^{(2')}) \\ &\stackrel{(1.9)}{=} \varphi^+(\mathcal{R}^{(1)})(a_2 \triangleleft \mathcal{R}^{(2)}) a_1 \\ &\stackrel{(1.13)}{=} \chi^+(a_2) a_1, \end{aligned}$$

which proves (1.15) in this case. Moreover

$$\begin{aligned} \chi^+(a_2 a'_2) &\stackrel{(1.13)}{=} \varphi_1^+(\mathcal{R}^{(1)})(a_2 a'_2 \triangleleft \mathcal{R}^{(2)}) \\ &\stackrel{(1.8)}{=} \varphi^+(\mathcal{R}^{(1)})(a_2 \triangleleft \mathcal{R}_{(1)}^{(2)})(a'_2 \triangleleft \mathcal{R}_{(2)}^{(2)}) \\ &\stackrel{(A.1.4)}{=} \varphi^+(\mathcal{R}^{(1')} \mathcal{R}^{(1)})(a_2 \triangleleft \mathcal{R}^{(2)})(a'_2 \triangleleft \mathcal{R}^{(2')}) \\ &\stackrel{(1.17)}{=} \varphi^+(\mathcal{R}^{(1')}) \varphi(\mathcal{R}^{(1)})(a_2 \triangleleft \mathcal{R}^{(2)})(a'_2 \triangleleft \mathcal{R}^{(2')}) \\ &\stackrel{(1.13)}{=} \varphi^+(\mathcal{R}^{(1')}) \chi^+(a_2)(a'_2 \triangleleft \mathcal{R}^{(2')}) \\ &\stackrel{(1.15)}{=} \chi^+(a_2) \varphi(\mathcal{R}^{(1')})(a'_2 \triangleleft \mathcal{R}^{(2')}) \\ &\stackrel{(1.13)}{=} \chi^+(a_2) \chi^+(a'_2), \end{aligned}$$

proving that χ^+ is a homomorphism. To prove injectivity we show that χ^+ can be inverted on $\chi^+(\mathcal{A}_2)$, and the inverse is given by

$$(\chi^+)^{-1}(\tilde{a}_2) = V^{-1} \left([\varphi^+(S^{-1}\mathcal{R}^{(1)})\tilde{a}_2] \triangleleft \mathcal{R}^{(2)} \right) \quad (1.19)$$

where $V \in \mathcal{A}_1$ is the invertible element defined by $V := \varphi_1^+(S^{-1}\mathcal{R}^{(1)}) \triangleleft \mathcal{R}^{(2)}$ (V is invertible because \mathcal{R} is). In fact,

$$\begin{aligned} & V^{-1} [\varphi_1^+(S^{-1}\mathcal{R}^{(1)})\chi^+(a_2)] \triangleleft \mathcal{R}^{(2)} \\ \stackrel{(1.13)}{=} & V^{-1} [\varphi_1^+(S^{-1}\mathcal{R}^{(1)})\varphi_1^+(\mathcal{R}^{(1')}) (a_2 \triangleleft \mathcal{R}^{(2')})] \triangleleft \mathcal{R}^{(2)} \\ \stackrel{(A.1.5),(1.17)}{=} & V^{-1} \left\{ \varphi_1^+[S^{-1}(\mathcal{R}^{-1(1')}\mathcal{R}^{(1)})] (a_2 \triangleleft \mathcal{R}^{-1(2')}) \right\} \triangleleft \mathcal{R}^{(2)} \\ \stackrel{(1.8)}{=} & V^{-1} \varphi_1^+[S^{-1}(\mathcal{R}^{-1(1')}\mathcal{R}^{(1)})] \triangleleft \mathcal{R}^{(2)}_{(1)} (a_2 \triangleleft \mathcal{R}^{-1(2')}) \triangleleft \mathcal{R}^{(2)}_{(2)} \\ \stackrel{(A.1.4)}{=} & V^{-1} \varphi_1^+[S^{-1}(\mathcal{R}^{-1(1')}\mathcal{R}^{(1)}\mathcal{R}^{(1'')})] \triangleleft \mathcal{R}^{(2'')} (a_2 \triangleleft \mathcal{R}^{-1(2')}) \triangleleft \mathcal{R}^{(2)} \\ \stackrel{(1.7)}{=} & V^{-1} \varphi_1^+[S^{-1}(\mathcal{R}^{(1'')})] \triangleleft \mathcal{R}^{(2'')} a_2 \\ = & V^{-1} V a_2 = a_2. \end{aligned}$$

In fact, if φ_1^+ can be extended to an algebra homomorphism $\varphi_1 : \mathcal{A}_1 \rtimes H \rightarrow \mathcal{A}_1$ a little calculation with the help of eq.'s (A.2.1), (A.1.7) shows that $V = \varphi_1(v)$, where $v \in H$ is the invertible central element defined by (A.1.8). We know that $\mathcal{A}_1 \otimes \tilde{\mathcal{A}}_2^+ \subset \mathcal{A}^+$. To prove that $\mathcal{A}^+ = \mathcal{A}_1 \otimes \tilde{\mathcal{A}}_2^+$ note first that by (1.9) any element in \mathcal{A}^+ can be written as a sum of products $a_1 a_2$, with $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$. So we need to show that

$$a_1 a_2 = b^{(1)} \chi^+(b^{(2)}) \quad (1.20)$$

for some $b^{(1)} \in \mathcal{A}_1$, $b^{(2)} \in \mathcal{A}_2$ (at the rhs a sum of many terms is implicitly understood). Now this can be proved as follows:

$$\begin{aligned} a_1 a_2 &= a_1 \varphi_1^+(\mathbf{1}_H) (a_2 \triangleleft \mathbf{1}_H) = a_1 \varphi_1^+(\mathcal{R}^{-1(1)}\mathcal{R}^{(1')}) [a_2 \triangleleft (\mathcal{R}^{-1(2)}\mathcal{R}^{(2')})] \\ &\stackrel{(1.7)}{=} a_1 \varphi_1^+(\mathcal{R}^{-1(1)}) \varphi_1^+(\mathcal{R}^{(1')}) (a_2 \triangleleft \mathcal{R}^{-1(2)}) \triangleleft \mathcal{R}^{(2')} \\ &\stackrel{(1.13)}{=} a_1 \varphi_1^+(\mathcal{R}^{-1(1)}) \chi^+(a_2 \triangleleft \mathcal{R}^{-1(2)}), \end{aligned}$$

which is of the form (1.20).

The proof for χ^- under the corresponding assumptions is completely analogous. \square

In the next Section we shall need an alternative expression for χ^\pm , which we prove in the appendix:

Proposition 1

$$\chi^+(a_2) = (a_2 \triangleleft \mathcal{R}^{-1(2)}) \varphi_1^+(S\mathcal{R}^{-1(1)}), \quad (1.21)$$

$$\chi^-(a_2) = (a_2 \triangleleft \mathcal{R}^{(1)}) \varphi_1^-(S\mathcal{R}^{(2)}). \quad (1.22)$$

The rest of the paper is essentially devoted to illustrate the application of Theorem 1 to some algebras \mathcal{A}_i for which homomorphisms φ_1^\pm are known. In Ref. [4] algebra homomorphisms φ_1^\pm have been found for (a slightly enlarged version \mathcal{A}_1 of) the algebra of functions on the N -dimensional quantum Euclidean space [12] \mathbb{R}_q^N , corresponding to $H = U_q\mathfrak{so}(N)$. The explicit forms of φ_1^\pm on the Faddeev-Reshetikhin-Takhtadjan (FRT) generators $\mathcal{L}_j^{\pm i}$ of $U_q\mathfrak{so}(N)$ are recalled in the appendix A.3. The maps φ_1^\pm for $N = 3$ are given also in Ref. [5]. The same maps do the job also on the quotient spaces obtained by setting $x^i x_i = 1$ [quantum $(N-1)$ -dimensional spheres S_q^{N-1}], and the appropriate maps for the q -deformed fuzzy sphere $S_{q,M}^2$ have been found in [19]. Therefore $U_q\mathfrak{so}(N)$ and the quantum Euclidean spaces/spheres provide nontrivial H and \mathcal{A}_1 for the application of the above theorem. In fact, the constructions of the frame given in Ref. [17, 4] can be interpreted as an application of the theorem with $\mathcal{A}_1 \equiv \mathbb{R}_q^N$ and \mathcal{A}_2 the $N!$ -dim exterior algebra generated by the differentials dx^i of the $U_q\mathfrak{so}(N)$ -covariant differential calculus (although with a universal R -matrix \mathcal{R} slightly modified by multiplication by the coproduct $\Delta(\Lambda) = \Lambda \otimes \Lambda$ of a new element Λ generating dilatations); consequently, in agreement with the philosophy of Ref. [25], the algebra of differential forms on \mathbb{R}_q^N can be written as $\mathbb{R}_q^N \otimes \tilde{\mathcal{A}}_2$, where $\tilde{\mathcal{A}}_2$ is the $N!$ -dim exterior algebra generated by the frame elements. On the other hand, the existence of algebra homomorphisms $\varphi : \mathcal{A}_1 \rtimes H \rightarrow \mathcal{A}_1$, for $H = U_q\mathfrak{so}(N), U_q\mathfrak{sl}(N)$ and \mathcal{A}_1 respectively equal to (a suitable completion of) the $U_q\mathfrak{so}(N)$ -covariant Heisenberg algebra or the $U_q\mathfrak{sl}(N)$ -covariant Heisenberg or Clifford algebras, has been known for even a longer time [13, 8, 21], so the theorem also applies if we choose as (H, \mathcal{A}_1) one of these pairs of algebras.

Of course the above theorem can be used iteratively to completely unbraided an algebra \mathcal{A} obtained by repeated braided tensor product [through prescription (1.9), or prescription (1.10)] of an arbitrary number of H -module algebras $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_M$. We shall explicitly consider the particular case that the latter be M identical copies of the $U_q\mathfrak{so}(N)$ -covariant quantum space/sphere (Section 3), of the $U_q\mathfrak{so}(3)$ -covariant q -fuzzy sphere (Section 6), or of the $U_q\mathfrak{so}(N)$ - or $U_q\mathfrak{sl}(N)$ -covariant Heisenberg algebra (Section 4). There we shall explicitly write down the generators of $\tilde{\mathcal{A}}_2^\pm$ for the lowest N examples.

In appendix A.3 we analyze the properties of φ^\pm under the main real sections of $U_q\mathfrak{so}(N)$, what was left aside in Ref. [4]. In section 2 we investigate in the context of general position the properties of χ^\pm under the $*$ -structures.

2 The unbraiding under the $*$ -structures

Assume H is a Hopf $*$ -algebra, namely the coproduct Δ and counit ε are $*$ -homomorphisms,

$$\Delta(g^*) \equiv (g^*)_{(1)} \otimes (g^*)_{(2)} = (g_{(1)})^* \otimes (g_{(2)})^*, \quad (2.1)$$

and $\mathcal{A}_1, \mathcal{A}_2$ are H -module $*$ -algebras, namely for any $a_i \in \mathcal{A}_i$

$$(a_i \triangleleft g)^* = a_i^* \triangleleft S^{-1}g^* \quad (2.2)$$

(here S denotes the antipode of H); we have used and shall use the same symbol $*$ for the $*$ -structure on all algebras H, \mathcal{A}_1 , etc. Then $*$ is a $*$ -structure also for $\mathcal{A}_1 \bowtie H$. The same statement is not automatically true for the braided tensor product algebra $\mathcal{A}^\pm = \mathcal{A}_1 \underline{\otimes}^\pm \mathcal{A}_2$, because the basic requirement that the latter be antimultiplicative

$$(a_2 a_1)^* = a_1^* a_2^* \quad (2.3)$$

(note that this would make \mathcal{A}^\pm also a H -module $*$ -algebra) is not automatically guaranteed. In fact, applying this would-be $*$ to rhs of (1.9) one finds

$$\begin{aligned} (a_2 a_1)^* &\stackrel{(1.9)}{=} \left[(a_1 \triangleleft \mathcal{R}^{(1)}) (a_2 \triangleleft \mathcal{R}^{(2)}) \right]^* \\ &\stackrel{(2.3)}{=} (a_2 \triangleleft \mathcal{R}^{(2)})^* (a_1 \triangleleft \mathcal{R}^{(1)})^* \\ &\stackrel{(2.2)}{=} (a_2^* \triangleleft S^{-1} \mathcal{R}^{(2)*}) (a_1^* \triangleleft S^{-1} \mathcal{R}^{(1)*}) \\ &\stackrel{(1.9)}{=} (a_1^* \triangleleft S^{-1} \mathcal{R}^{(1)*} \mathcal{R}^{(1')}) (a_2^* \triangleleft S^{-1} \mathcal{R}^{(2)*} \mathcal{R}^{(2')}); \end{aligned} \quad (2.4)$$

in order that this be equal to the rhs of (2.3) it is necessary that $(S^{-1} \otimes S^{-1}) \mathcal{R}^* = \mathcal{R}^{-1}$, which upon use of (A.1.5) is equivalent to

$$\mathcal{R}^* = \mathcal{R}^{-1} \quad (2.5)$$

(here \mathcal{R}^* means $\mathcal{R}^{(1)*} \otimes \mathcal{R}^{(2)*}$). This condition is fulfilled only for the standard noncompact sections (A.1.28) of $U_q \mathfrak{g}$, for $|q| = 1$; as a consequence, $\mathcal{A}^+ = \mathcal{A}_1 \underline{\otimes}^+ \mathcal{A}_2$ becomes a H -module $*$ -algebra if one extends the $*$ -structures of the tensor factors to \mathcal{A} using (2.3). The same holds for \mathcal{A}^- .

On the contrary, the compact section, which requires $q \in \mathbb{R}$, is characterized by

$$\mathcal{R}^* = \mathcal{R}_{21}. \quad (2.6)$$

In the latter case the map $*$ introduced through (2.3) makes sense only as an involutive antimultiplicative antilinear map $\mathcal{A}^+ \rightarrow \mathcal{A}^-$, if both \mathcal{A}^+ and \mathcal{A}^- exist. In fact, in this case the last line in (2.4) will be replaced by

$$\stackrel{(1.10), (2.6)}{=} (a_1^* \triangleleft S^{-1} \mathcal{R}^{(2)} \mathcal{R}^{-1(2')}) (a_2^* \triangleleft S^{-1} \mathcal{R}^{(1)} \mathcal{R}^{-1(1')}) \stackrel{(A.1.5)}{=} a_1^* a_2^*,$$

as required. Alternatively, if $\mathcal{A}_1, \mathcal{A}_2$ are two copies of the same algebra and we denote by $\psi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ the map associating to each $a_1 \in \mathcal{A}_1$ the equivalent element in \mathcal{A}_2 , one can define an alternative $*$ -structure \star in \mathcal{A}^\pm by setting

$$a_1^\star = \psi(a_1^*) \quad a_2^\star = \psi^{-1}(a_2^*), \quad (2.7)$$

since this is instead compatible with (1.9). In fact, (2.4) will become

$$\begin{aligned} (a_2 a_1)^\star &\stackrel{(1.9)}{=} \left[(a_1 \triangleleft \mathcal{R}^{(1)}) (a_2 \triangleleft \mathcal{R}^{(2)}) \right]^\star \\ &\stackrel{(2.3)}{=} (a_2 \triangleleft \mathcal{R}^{(2)})^\star (a_1 \triangleleft \mathcal{R}^{(1)})^\star \\ &\stackrel{(2.2)}{=} \psi^{-1}(a_2^* \triangleleft S^{-1} \mathcal{R}^{(2)*}) \psi(a_1^* \triangleleft S^{-1} \mathcal{R}^{(1)*}) \\ &\stackrel{(1.9)}{=} \psi(a_1^* \triangleleft S^{-1} \mathcal{R}^{(2)} \mathcal{R}^{-1(2')}) \psi^{-1}(a_2^* \triangleleft S^{-1} \mathcal{R}^{(1)} \mathcal{R}^{-1(1')}) \\ &\stackrel{(A.1.5)}{=} \psi(a_1^*) \psi^{-1}(a_2^*) a_1 \\ &\stackrel{(2.7)}{=} a_1^\star a_2^\star \end{aligned} \quad (2.8)$$

A similar trick can be used also if one considers an iterated braided tensor product of $M > 2$ copies of the same algebra, see next section. However, such \star 's have not the standard commutative limit, because of the presence of the map ψ .

Inspired by the applications of the next two Sections, we now assume that φ_1^\pm fulfill some specific conditions relating its action before and after the application of the involution $*$, and analyze the identities relating the action of χ^\pm before and after the application of $*$ which follow herefrom.

Proposition 2 *Assume that the conditions of Theorem 1 for defining χ^+ or χ^- are fulfilled. If $\mathcal{R}^* = \mathcal{R}^{-1}$ and for any $g^\pm \in H^\pm$*

$$[\varphi_1^\pm(g^\pm)]^* = \varphi_1^\pm(g^{\pm*}), \quad (2.9)$$

in other words φ_1^\pm are $$ -homomorphisms, then*

$$[\chi^\pm(a_2)]^* = \chi^\pm(a_2^*). \quad (2.10)$$

If $\mathcal{R}^ = \mathcal{R}_{21}$ and $*$: $H^\pm \rightarrow H^\mp$ fulfills*

$$[\varphi_1^\pm(g)]^* = \varphi_1^\mp(g^*), \quad (2.11)$$

then

$$[\chi^\pm(a_2)]^* = \chi^\mp(a_2^*). \quad (2.12)$$

Proof . Under the first assumptions, for any $a_2 \in \mathcal{A}_2$,

$$[\chi^+(a_2)]^* \stackrel{(1.21)}{=} [(a_2 \triangleleft \mathcal{R}^{-1(2)}) \varphi_1^+(S \mathcal{R}^{-1(1)})]^*$$

$$\begin{aligned}
&= [\varphi_1^+(S\mathcal{R}^{-1(1)})]^*(a_2 \triangleleft \mathcal{R}^{-1(2)})^* \\
&\stackrel{(2.9),(2.2)}{=} \varphi_1^+(S^{-1}\mathcal{R}^{-1(1)*})(a_2^* \triangleleft S^{-1}\mathcal{R}^{-1(2)*}) \\
&\stackrel{(2.5)}{=} \varphi_1^+(S^{-1}\mathcal{R}^{(1)})(a_2^* \triangleleft S^{-1}\mathcal{R}^{(2)}) \\
&\stackrel{(A.1.5)}{=} \varphi_1^+(\mathcal{R}^{(1)})(a_2^* \triangleleft \mathcal{R}^{(2)}) \\
&\stackrel{(1.13)}{=} \chi^+(a_2^*).
\end{aligned}$$

Similarly one proves (2.10) for χ^- . Under the second assumptions, for any $a_2 \in \mathcal{A}_2$,

$$\begin{aligned}
[\chi^+(a_2)]^* &\stackrel{(1.21)}{=} [(a_2 \triangleleft \mathcal{R}^{-1(2)}) \varphi_1^+(S\mathcal{R}^{-1(1)})]^* \\
&= [\varphi_1^+(S\mathcal{R}^{-1(1)})]^*(a_2 \triangleleft \mathcal{R}^{-1(2)})^* \\
&\stackrel{(2.11),(2.2)}{=} \varphi_1^-(S^{-1}\mathcal{R}^{-1(1)*})(a_2^* \triangleleft S^{-1}\mathcal{R}^{-1(2)*}) \\
&\stackrel{(2.6)}{=} \varphi_1^-(S^{-1}\mathcal{R}^{-1(2)})(a_2^* \triangleleft S^{-1}\mathcal{R}^{-1(1)}) \\
&\stackrel{(A.1.5)}{=} \varphi_1^-(\mathcal{R}^{-1(2)})(a_2^* \triangleleft \mathcal{R}^{-1(1)}) \\
&\stackrel{(1.14)}{=} \chi^-(a_2^*).
\end{aligned}$$

By similar arguments one proves the claim for χ^- . \square

It should be noted that there also exist non-standard star structures on $U_q\mathfrak{g}$ for $|q| = 1$, in particular the compact form $X_i^{\pm*} = X_i^{\mp}$, $K_i^* = K_i^{-1}$ in terms of the Cartan–Weyl generators. Then

$$\mathcal{R}^* = \mathcal{R}_{21}^{-1}, \quad (2.13)$$

while the coproduct does not fulfill (2.1) as in a standard Hopf $*$ -algebra but becomes flipped under the star. This nevertheless has the correct classical limit, because the coproduct is cocommutative for $q = 1$. In certain cases (in particular on the fuzzy quantum sphere [19] discussed in section 6, but see also [30]), it is then possible to define a star structure on each \mathcal{A}_i , which takes the form $a_{i;k}^* = \pm \Omega_i a_{i;k} \Omega_i^{-1}$ on the generators $a_{i;k}$ of \mathcal{A}_i . Here $\Omega_i = \sqrt[4]{v_i^{-1}} \omega_i$, where v_i and ω_i are the realizations in \mathcal{A}_i (using an algebra map from $U_q\mathfrak{g}$ to \mathcal{A}_i as above) of the central element $v \in U_q\mathfrak{g}$ (A.1.8) and the “universal Weyl element” ω in an extension of $U_q\mathfrak{g}$ [22]. All this must be defined in some representation of \mathcal{A}_i ; for more details see* [19, 30]. If moreover there exists an element Ω which realizes $\sqrt[4]{v}^{-1} \omega$ in $\mathcal{A}^+ = \mathcal{A}_1 \otimes^+ \mathcal{A}_2$ or a “physical” subspace thereof, then it follows easily from (2.13) that the star structure $a_{i;k}^* = \pm \Omega a_{i;k} \Omega^{-1}$ on \mathcal{A}^+ is consistent with the commutation relations of the braided tensor product algebra \mathcal{A}^+ . This star then has the correct classical limit, and the same construction also works for \mathcal{A}^- .

*The v in [19, 30] is the square root of our v here.

3 Unbraiding ‘chains’ of braided quantum Euclidean spaces or spheres

In this section we consider the braided tensor product of $M \geq 2$ copies $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_M$ of the quantum Euclidean space \mathbb{R}_q^N [12] (the $U_q\mathfrak{so}(N)$ -covariant quantum space), i.e. of the unital associative algebra generated by x^i fulfilling the relations

$$\mathcal{P}_{a_{hk}}^{ij} x^h x^k = 0, \quad (3.1)$$

where \mathcal{P}_a denotes the q -deformed antisymmetric projector appearing in the decomposition of the braid matrix \hat{R} of $U_q\mathfrak{so}(N)$ [given in formula (A.1.23)], or of the quotient space of the latter obtained by setting $r^2 := x^i x_i = 1$ [the quantum $(N-1)$ -dimensional sphere S_q^{N-1}]. The multiplet (x^i) carries the fundamental vector representation ρ of $U_q\mathfrak{so}(N)$: for any $g \in U_q\mathfrak{so}(N)$

$$x^i \triangleleft g = \rho_j^i(g) x^j. \quad (3.2)$$

We shall enumerate the different copies of the quantum Euclidean space or sphere by attaching an additional greek index to them, e.g. $\alpha = 1, 2, \dots, M$. The prescription (1.10) to glue $\mathcal{A}_1, \dots, \mathcal{A}_M$ into a $U_q\mathfrak{so}(N)$ -module associative algebra \mathcal{A}^- gives the following cross commutation relations between their respective generators:

$$x^{\alpha,i} x^{\beta,j} = \hat{R}_{hk}^{ij} x^{\alpha,h} x^{\beta,k} \quad (3.3)$$

whenever $\alpha < \beta$. Note that prescriptions (1.10), (1.9) go into each other under the inverse reordering $1, 2, \dots, M \rightarrow M, \dots, 2, 1$. Applying iteratively Theorem 1 we shall be able to completely unbraid this iterated tensor product.

To define φ_1^\pm one actually needs a slightly enlarged version [4] of \mathbb{R}_q^N (or S_q^{N-1}). One has to introduce some new generators $\sqrt{r_a}$, with $1 \leq a \leq \frac{N}{2}$, together with their inverses $(\sqrt{r_a})^{-1}$, requiring that

$$r_a^2 = \sum_{h=-a}^a x^h x_h = \sum_{h=-a}^a g_{hk} x^h x^k \quad (3.4)$$

(note that, having set $n := \lfloor \frac{N}{2} \rfloor$, r_n^2 coincides with r^2). Moreover for odd N we add also $\sqrt{x^0}$ and its inverse as new generators). In fact, the commutation relations involving these new generators can be fixed consistently, and turn out to be simply q -commutation relations. r plays the role of ‘deformed Euclidean distance’ of the generic ‘point of coordinates’ (x^i) of \mathbb{R}_q^N from the ‘origin’; r_a is the ‘projection’ of r on the ‘subspace’ $x^i = 0$, $|i| > a$. In the previous equation g_{hk} denotes the ‘metric matrix’ of $SO_q(N)$:

$$g_{ij} = g^{ij} = q^{-\rho_i} \delta_{i,-j}. \quad (3.5)$$

It is a $SO_q(N)$ -isotropic tensor and is a deformation of the ordinary Euclidean metric. Here and in the sequel $n := \lfloor \frac{N}{2} \rfloor$ is the rank of $so(N)$, the indices take the values $i = -n, \dots, -1, 0, 1, \dots, n$ for N odd, and $i = -n, \dots, -1, 1, \dots, n$ for N even. Moreover, we have introduced the notation $(\rho_i) = (n - \frac{1}{2}, \dots, \frac{1}{2}, 0, -\frac{1}{2}, \dots, \frac{1}{2} - n)$ for N odd, $(n - 1, \dots, 0, 0, \dots, 1 - n)$ for N even. In the case of even N one needs to include also the FRT generator \mathcal{L}^{-1}_1 and its inverse \mathcal{L}^{+1}_1 (which are generators of $U_q so(N)$ belonging to the Cartan subalgebra) among the generators of \mathcal{A}_1 . They satisfy the commutation relations

$$\mathcal{L}^{-1}_1 x^{\pm 1} = q^{\pm 1} x^{\pm 1} \mathcal{L}^{-1}_1, \quad \mathcal{L}^{-1}_1 x^{\pm i} = x^{\pm i} \mathcal{L}^{-1}_1 \text{ for } i > 1. \quad (3.6)$$

with \mathcal{A}_1 , and the standard FRT relations with the rest of $U_q so(N)$. One can easily show that the extension of the action of $U_q so(N)$ to $\sqrt{r_a}, (\sqrt{r_a})^{-1}$ is uniquely determined by the constraints the latter fulfil; it is a bit complicated and therefore will be omitted, since we will not need its explicit expression. The action of H on \mathcal{L}^{-1}_1 is the standard (right) adjoint action. Note that the maps φ_1^{\pm} have no analog in the “undeformed” case ($q = 1$), because $\mathcal{A}_1 \equiv \mathbb{R}^N$ is abelian, whereas $H \equiv U_q so(N)$ is not.

The unbraiding procedure is recursive. We use the homomorphism φ_1 found in Ref. [4] and start by unbraiding the first copy from the others. Following Theorem 1, we perform the following change of generators in \mathcal{A}^-

$$y^{1,i} := x^{1,i} \\ y^{\alpha,i} := \chi^-(x^{\alpha,i}) \stackrel{(1.14)}{=} \varphi_1(\mathcal{R}^{-1(2)}) \rho_j^i (\mathcal{R}^{-1(1)}) x^{\alpha,j} = \varphi_1(\mathcal{L}^{-i}_j) x^{\alpha,j}, \quad \alpha > 1.$$

In the last equality we have used the definition (A.1.15) of the FRT generators [12] of $U_q so(N)$. In appendix A.3 we recall the φ^{\pm} images of the latter. In view of formula (A.3.2) we thus find

$$y^{1,i} := x^{1,i} \\ y^{\alpha,i} := g^{ih} [\mu_h^1, x^{1,k}]_q g_{kj} x^{\alpha,j}, \quad \alpha > 1. \quad (3.7)$$

The suffix 1 in μ_a^1 means that the special elements μ_a defined in (A.3.3) must be taken as elements of the first copy of \mathbb{R}_q^N . In view of (A.3.3) we see that $g^{ih} [\mu_h^1, x^{1,k}]_q g_{kj}$ are rather simple polynomials in x^i and r_a^{-1} , homogeneous of total degree 1 in the coordinates x^i and r_a . Hence (3.7) is a transformation of polynomial type and therefore likely to be implemented as a well-defined operator transformation also when representing \mathcal{A}^- as an algebra of operators on some linear space. Using the results (A.3.8) given in the appendix we give now the explicit expression of (3.7) for $N = 3$:

$$y^{\alpha,-} = -qh\gamma_1 \frac{r}{x^0} x^{\alpha,-} \\ y^{\alpha,0} = \sqrt{q}(q+1) \frac{1}{x^0} x^+ x^{\alpha,-} + x^{\alpha,0} \\ y^{\alpha,+} = \frac{\sqrt{q}(q+1)}{h\gamma_1 r x^0} (x^+)^2 x^{\alpha,-} + \frac{q^{-1}+1}{h\gamma_1 r} x^+ x^{\alpha,0} - \frac{1}{qh\gamma_1 r} x^0 x^{\alpha,+} \quad (3.8)$$

for any $\alpha = 2, \dots, M$. Here we have set $x^i \equiv x^{1,i}$, $h \equiv \sqrt{q} - 1/\sqrt{q}$, replaced for simplicity the values $-1, 0, 1$ of the indices by the ones $-, 0, +$ and denoted by $\gamma_1 \in \mathbb{C}$ a free parameter.

As a consequence of the theorem we find

Corollary 1

$$[y^{1,i}, y^{\alpha,j}] = 0 \quad \alpha > 1 \quad (3.9)$$

$$y^{\alpha,i} y^{\beta,j} = \hat{R}_{hk}^{ij} y^{\alpha,h} y^{\beta,k} \quad 1 < \alpha < \beta \quad (3.10)$$

$$\mathcal{P}_{ahk}^{ij} y^{\alpha,h} y^{\alpha,k} = 0 \quad (3.11)$$

By (3.9) the subalgebra $\tilde{\mathcal{A}}_1^- \equiv \mathcal{A}_1^-$ of \mathcal{A}^- generated by $y^{1,i} \equiv x^{1,i}$ commutes with the subalgebra generated by $y^{2,i}, \dots, y^{M,i}$, which we shall call $\tilde{\mathcal{A}}^-$. This was the first step of the unbraiding procedure. Now we can reiterate the latter for $\tilde{\mathcal{A}}^-$, with $y^{2,i}$ playing the role of $x^{1,i}$. After $M - 1$ steps, we shall have determined M independent commuting subalgebrae of \mathcal{A}^- which we shall call $\tilde{\mathcal{A}}_\alpha^-$, $\alpha = 1, \dots, M$.

The unbraiding procedure for the alternative braided tensor product stemming from prescription (1.9) arises by iterating the change of generators

$$\begin{aligned} y'^{M,i} &:= x^{M,i} \\ y'^{\alpha,i} &:= \varphi_M(\mathcal{L}^{+i}_j) x^{\alpha,j} = g^{ih} [\bar{\mu}_h^M, x^{M,k}]_{q^{-1}} g_{kj} x^{\alpha,j}, \quad \alpha < M. \end{aligned} \quad (3.12)$$

$\bar{\mu}_a^M$ are the special elements defined in (A.3.6) belonging to the M -th copy of \mathbb{R}_q^N . Using the results (A.3.9) given in the appendix we give the explicit expression of (3.12) for $N = 3$: for any $\alpha = 1, \dots, M - 1$,

$$\begin{aligned} y'^{\alpha,-} &= -h\bar{\gamma}_1 \frac{z^0}{r_z} x^{\alpha,-} + \frac{k\bar{\gamma}_1}{\sqrt{q}r_z} z^- x^{\alpha,0} + \frac{q^{-2}k\bar{\gamma}_1}{r_z z^0} (z^-)^2 x^{\alpha,+} \\ y'^{\alpha,0} &= x^{\alpha,0} + q^{-\frac{1}{2}}(q^{-1} + 1) \frac{1}{z^0} z^- x^{\alpha,+} \\ y'^{\alpha,+} &= -\frac{r_z}{h\bar{\gamma}_1 z^0} x^{\alpha,+} \end{aligned} \quad (3.13)$$

Here we have set $z^i \equiv x^{M,i}$, $r_z^2 \equiv x^{M,i} x_i^M$, $k \equiv q - q^{-1}$ and $\bar{\gamma}_1 \in \mathbb{C}$ is a free parameter.

Again, the subalgebra $\tilde{\mathcal{A}}_M^+ \approx \mathbb{R}_q^N$ of \mathcal{A}^+ generated by $y^{M,i} \equiv x^{M,i}$ commutes with the subalgebra generated by $y^{1,i}, \dots, y^{M-1,i}$, which we shall call $\tilde{\mathcal{A}}^+$. This was the first step of the unbraiding procedure. Now we can reiterate the latter for $\tilde{\mathcal{A}}^+$, with $y^{M-1,i}$ playing the role of $x^{M,i}$. After $M - 1$ steps, we shall have determined M independent commuting subalgebrae of \mathcal{A}^+ which we shall call $\tilde{\mathcal{A}}_\alpha^+$.

We summarize the results of this section:

Proposition 3 *Let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_M$ be M copies of the $U_q so(N)$ -covariant quantum Euclidean space (or sphere). Then $\mathcal{A}_1 \underline{\otimes}^\pm \mathcal{A}_2 \underline{\otimes}^\pm \dots \underline{\otimes}^\pm \mathcal{A}_M = \mathcal{A}_1 \otimes \tilde{\mathcal{A}}_2^\pm \otimes \dots \otimes \tilde{\mathcal{A}}_M^\pm$, where $\tilde{\mathcal{A}}_2^\pm, \dots, \tilde{\mathcal{A}}_M^\pm$ are subalgebrae of the lhs isomorphic to \mathcal{A}_1 .*

By a suitable choice of $\gamma_1, \bar{\gamma}_1$, as well as of the other free parameters appearing in the definitions of φ^\pm for $N > 3$ (see appendix A.3), one can make φ^\pm into $*$ -homomorphisms when $|q| = 1$, and make them satisfy the relation

$$[\varphi^\pm(g)]^* = \varphi^\mp(g^*) \quad (3.14)$$

when $q \in \mathbb{R}^+$. Since these relations are of the type considered in proposition 2, the claims of the latter for χ^\pm and their consequences hold. In particular, when $|q| = 1$ one has a well-defined $*$ on the braided tensor product of $\mathcal{A}_1, \dots, \mathcal{A}_M$ mapping each of the independent, commuting subalgebras $\tilde{\mathcal{A}}_i^\pm$ into itself. On the contrary for real q one can consider the map $*$: $\mathcal{A}^+ \rightarrow \mathcal{A}^-$ defined by (2.3) or a $*$ -structure on \mathcal{A}^\pm defined in a way similar to what we have done in (2.7),

$$(x^{\alpha,i})^* = x^{M-\alpha+1,j} g_{ji}. \quad (3.15)$$

The latter has not the standard classical limit. A short calculation shows that the latter implies

$$(y^{\alpha,i})^* = y'^{M-\alpha+1,j} g_{ji}. \quad (3.16)$$

4 Unbraiding ‘chains’ of braided Heisenberg algebras

In this section we consider the braided tensor product of $M \geq 2$ copies $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_M$ of the $U_q \mathfrak{g}$ -covariant deformed Heisenberg algebra $\mathcal{D}_{\epsilon, \mathfrak{g}}$, $\mathfrak{g} = sl(N), so(N)$ [28, 31, 3], i.e. the unital associative algebra generated by x^i, ∂_j fulfilling the relations

$$\begin{aligned} \mathcal{P}_{ahk}^{ij} x^h x^k &= 0 \\ \mathcal{P}_{ahk}^{ij} \partial_j \partial_i &= 0 \\ \partial_i x^j &= \delta_j^i + (q\gamma \hat{R})_{ih}^{\epsilon jk} x^h \partial_k, \end{aligned} \quad (4.1)$$

where $\gamma = q^{\frac{1}{N}}, 1$ respectively for $\mathfrak{g} = sl(N), so(N)$, and the exponent ϵ can take either value $\epsilon = 1, -1$. \hat{R} denotes the braid matrix of $U_q \mathfrak{g}$ [given in formulae (A.1.22) and (A.1.23)], and again \mathcal{P}_a the antisymmetric projector appearing in the decomposition of the latter. The coordinates x^i transform according to the fundamental vector representation of $U_q \mathfrak{g}$, as in (3.2), whereas the ‘partial derivatives’ transform according the contragredient representation,

$$\partial_i \triangleleft g = \partial_h \rho_i^h (S^{-1} g). \quad (4.2)$$

The indices will take the values $i = 1, \dots, N$ if $\mathfrak{g} = sl(N)$, the same values considered in the previous section if $\mathfrak{g} = so(N)$. Clearly in the latter

case $\mathcal{D}_{\epsilon, \mathbf{g}}$ has the quantum Euclidean space generated by x^i as a module subalgebra.

Again, we shall enumerate the different copies by attaching to them an additional greek index, e.g. $\alpha = 1, 2, \dots, M$. The prescription (1.9) to glue $\mathcal{A}_1, \dots, \mathcal{A}_M$ into a $U_q \mathbf{g}$ -module associative algebra \mathcal{A}^+ (see also Ref. [14]) gives the following cross commutation relations between their respective generators

$$\begin{aligned} x^{\alpha, i} x^{\beta, j} &= \hat{R}_{hk}^{ij} x^{\beta, h} x^{\alpha, k} & \partial_{\alpha, i} \partial_{\beta, j} &= \hat{R}_{ji}^{kh} \partial_{\beta, h} \partial_{\alpha, k} \\ \partial_{\alpha, i} x^{\beta, j} &= \hat{R}_{ik}^{-1jh} x^{\beta, k} \partial_{\alpha, h} & \partial_{\beta, i} x^{\alpha, j} &= \hat{R}_{ik}^{jh} x^{\alpha, k} \partial_{\beta, h} \end{aligned} \quad (4.3)$$

when $\alpha > \beta$. With respect to Ref. [14] we have called the generators x^i, ∂_j instead of A^i, A_j^+ , inverted the order of the product due to covariance w.r.t. the *right* (instead of the *left*) $U_q \mathbf{g}$ -action, and for the sake of simplicity we have put equal to one possible factors at the rhs of (4.3).

In Ref. [13, 8] algebra homomorphisms $\varphi : \mathcal{D}_{\epsilon, \mathbf{g}} \rtimes H \rightarrow \mathcal{D}_{\epsilon, \mathbf{g}}$ have been determined for $\mathbf{g} = so(n)$ and $\mathbf{g} = sl(N), so(N)$ respectively. This is the q -analog of vector field realization of \mathbf{g} on the corresponding \mathbf{g} -covariant (undeformed) space, e.g. $\varphi_1(E_j^i) = x^i \partial_j - \frac{1}{N} \delta_j^i$ in the $\mathbf{g} = sl(N)$ case. The searched maps φ^\pm will be simply the restrictions of φ to $\mathcal{D}_{\epsilon, \mathbf{g}} \rtimes H^\pm$. In Ref. [13] there are among others the φ -images of the Chevalley generators of $U_q so(N)^\dagger$, in Ref. [8] there are the φ -images of the generators of $U_q \mathbf{g}$ playing the role of “vector fields” on G_q . By the change of generators described in Ref. [12] one can easily pass from the Chevalley to the FRT generators $\mathcal{L}_j^{\pm i}$ (A.1.15), whereas the relation between the latter and the vector fields is recalled in (A.4.2). The FRT generators are the ones explicitly needed in writing down $\chi^\pm(x^i)$ and $\chi^\pm(\partial_i)$. For example, for $\mathbf{g} = sl(2)$ and $\epsilon = 1$ one finds

$$\begin{aligned} \varphi(\mathcal{L}_1^{+1}) &= \varphi(\mathcal{L}_2^{-2}) = [\varphi(\mathcal{L}_1^{-1})]^{-1} = [\varphi(\mathcal{L}_2^{+2})]^{-1} = \alpha \Lambda^{\frac{1}{2}} [1 + (q^2 - 1)x^2 \partial_2]^{\frac{1}{2}} \\ \varphi(\mathcal{L}_2^{+1}) &= -\alpha k q^{-1} \Lambda^{\frac{1}{2}} [1 + (q^2 - 1)x^2 \partial_2]^{-\frac{1}{2}} x^1 \partial_2 \\ \varphi(\mathcal{L}_1^{-2}) &= \alpha k q^3 \Lambda^{\frac{1}{2}} [1 + (q^2 - 1)x^2 \partial_2]^{-\frac{1}{2}} x^2 \partial_1, \end{aligned} \quad (4.4)$$

where α is fixed by (A.1.14) to be $\alpha = \pm 1, \pm i$ and we have set

$$\Lambda^{-2} := 1 + (q^2 - 1)x^i \partial_i. \quad (4.5)$$

Whereas for $\mathbf{g} = so(3)$ and $\epsilon = 1$ one finds on the positive Borel subalgebra

$$\begin{aligned} \varphi(\mathcal{L}^{+-}) &= -\alpha \Lambda [1 + (q - 1)x^0 \partial_0 + (q^2 - 1)x^+ \partial_+] \\ \varphi(\mathcal{L}_0^{+-}) &= \alpha k \Lambda (x^- \partial_0 - \sqrt{q} x^0 \partial_+) \\ \varphi(\mathcal{L}_+^{+-}) &= \frac{1}{1+q^{-1}} \varphi(\mathcal{L}_0^{+-}) \varphi(\mathcal{L}_+^{+0}) \\ \varphi(\mathcal{L}_0^{+0}) &= 1 \\ \varphi(\mathcal{L}_+^{+0}) &= -q^{-\frac{1}{2}} [\varphi(\mathcal{L}^{+-})]^{-1} \varphi(\mathcal{L}_0^{+-}) \\ \varphi(\mathcal{L}_+^{++}) &= [\varphi(\mathcal{L}^{+-})]^{-1} \end{aligned} \quad (4.6)$$

[†]One should take care of the fact that in Ref. [13] we considered $U_q so(N)$ acting by a *left* action, instead of a right one, what manifests itself in a replacement $q \rightarrow q^{-1}$, or equivalently in an opposite coproduct. The rules for passing from right to left are described in Sect. 5.

and on the negative Borel subalgebra

$$\begin{aligned}
\varphi(\mathcal{L}^{--}) &= -(\alpha\Lambda[1 + (q-1)x^0\partial_0 + (q^2-1)x^+\partial_+])^{-1} \\
\varphi(\mathcal{L}^{0-}) &= -\alpha q^2 k \varphi(\mathcal{L}^{--}) \Lambda(x^0\partial_- - \sqrt{q}x^+\partial_0) \\
\varphi(\mathcal{L}^{-+}) &= \frac{1}{1+q} \varphi(\mathcal{L}^{-+}_0) \varphi(\mathcal{L}^{0-}) \\
\varphi(\mathcal{L}^{00}) &= 1 \\
\varphi(\mathcal{L}^{-+}_0) &= -\alpha q^{\frac{3}{2}} k \Lambda(x^0\partial_- - \sqrt{q}x^+\partial_0) \\
\varphi(\mathcal{L}^{++}) &= [\varphi(\mathcal{L}^{--})]^{-1}.
\end{aligned} \tag{4.7}$$

Here we have set

$$\Lambda^{-2} := [1 + (q^2 - 1)x^i\partial_i + \frac{(q^2 - 1)^2}{\omega_1^2}(g_{ij}x^i x^j)(g^{hk}\partial_k\partial_h)], \tag{4.8}$$

where

$$\omega_a := (q^{\rho_a} + q^{-\rho_a}),$$

and replaced for simplicity the values $-1, 0, 1$ of the indices by the ones $-, 0, +$. In either case the φ -images of \mathcal{L}^{+i}_j and \mathcal{L}^{-j}_i for $i > j$ vanish, because the latter do.

We see that strictly speaking φ takes values in some appropriate completion of $\mathcal{D}_{\epsilon, \mathbf{g}}$, containing at least the square root and inverse square root of the polynomial Λ^{-2} respectively defined in (4.5), (4.8), as well as the square root of $[1 + (q^2 - 1)x^2\partial_2]$ and its inverse, when $\mathbf{g} = sl(2)$, and the inverses (4.6)₆, (4.7)₆, when $\mathbf{g} = so(3)$. Apart from this minimal completion, another possible one is the so-called h -adic, namely the ring of formal power series in $h = \log q$ with coefficients in $\mathcal{D}_{\epsilon, \mathbf{g}}$. Other completions, e.g. in operator norms, can be considered according to the needs. One can easily show that the extension of the action of H to any such completion is uniquely determined (we omit to write down its explicit expression, since we don't need it).

According to the main theorem, we set

$$\begin{aligned}
y^{1,i} &\equiv x^{1,i} \\
\partial_{y,1,a} &\equiv \partial_{1,a} \\
y^{\alpha,i} &\equiv \chi^-(x^{\alpha,i}) = \varphi_1(\mathcal{L}^{-i}_j)x^{\alpha,j} & \alpha > 1 \\
\partial_{y,\alpha,a} &\equiv \chi^-(\partial_{\alpha,a}) = \varphi_1(S\mathcal{L}^{-d}_a)\partial_{\alpha,d} & \alpha > 1
\end{aligned} \tag{4.9}$$

and we find

Corollary 2

$$\begin{aligned}
\mathcal{P}_{a_{hk}}^{ij} y^{\alpha,h} y^{\alpha,k} &= 0 \\
\mathcal{P}_{a_{ij}}^{hk} \partial_{y,\alpha,k} \partial_{y,\alpha,h} &= 0 \\
\partial_{y,\alpha,i} y^{\alpha,j} &= \delta_i^j + (q\hat{R})^{\epsilon_{\alpha} j l}_{im} y^{\alpha,m} \partial_{y,\alpha,l}
\end{aligned} \tag{4.10}$$

for all $\alpha = 1, \dots, M$, together with

$$\begin{aligned}
[y^{1,i}, y^{\alpha,j}] &= 0 & [\partial_{y,1,i}, y^{\alpha,j}] &= 0 \\
[\partial_{y,\alpha,i}, y^{1,j}] &= 0 & [\partial_{y,1,i}, \partial_{y,\alpha,j}] &= 0
\end{aligned} \tag{4.11}$$

when $\alpha > 1$, and

$$\begin{aligned} y^{\alpha,i} y^{\beta,j} &= \hat{R}_{hk}^{ij} y^{\beta,h} y^{\alpha,k} \\ \partial_{y,\alpha,i} \partial_{y,\beta,j} &= \hat{R}_{ji}^{kh} \partial_{y,\beta,h} \partial_{y,\alpha,k} \\ \partial_{y,\alpha,i} y^{\beta,j} &= \hat{R}_{ik}^{-1j}{}^h y^{\beta,k} \partial_{y,\alpha,h} \\ \partial_{y,\beta,i} y^{\alpha,j} &= \hat{R}_{ik}^{jh} y^{\alpha,k} \partial_{y,\beta,h} \end{aligned} \quad (4.12)$$

when $1 < \beta < \alpha$.

By (4.11) $y^{1,i} \equiv x^{1,i}$ and $\partial_{y,1,i} \equiv \partial_{1,i}$ commute with the subalgebra generated by $y^{2,i}, \dots, y^{M,i}$ and $\partial_{y,2,i}, \dots, \partial_{y,M,i}$ which we shall call $\tilde{\mathcal{A}}^+$. This was the first step of the unbraiding procedure. Now we can reiterate the latter for $\tilde{\mathcal{A}}^+$, with $y^{2,i}, \partial_{y,2,i}$ playing the role of $x^{1,i}, \partial_{1,i}$. After $M - 1$ steps, we shall have determined M independent commuting subalgebras of \mathcal{A}^+ which we shall call $\tilde{\mathcal{A}}_\alpha^+$.

For the sake of brevity we omit the unbraiding procedure for the alternative braided tensor product algebra stemming from prescription (1.10), which can be found following arguments completely analogous to the ones presented at the end of Section 3. We summarize the results of this section by

Proposition 4 *Let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_M$ be M copies of the $U_q \mathfrak{g}$ -covariant Heisenberg algebra $\mathcal{D}_{\epsilon, \mathfrak{g}}$, $\mathfrak{g} = sl(N), so(N)$. Then $\mathcal{A}_1 \underline{\otimes}^\pm \mathcal{A}_2 \underline{\otimes}^\pm \dots \underline{\otimes}^\pm \mathcal{A}_M = \mathcal{A}_1 \otimes \tilde{\mathcal{A}}_2^\pm \otimes \dots \otimes \tilde{\mathcal{A}}_M^\pm$, where $\tilde{\mathcal{A}}_2^\pm, \dots, \tilde{\mathcal{A}}_M^\pm$ are subalgebras of the lhs isomorphic to $\mathcal{D}_{\epsilon, \mathfrak{g}}$.*

Relations (A.1.28), (2.2), (3.2), (4.2) and (4.1) fix the $*$ -structure of \mathcal{A}_1 to be

$$(x^i)^* = x^i, \quad (\partial_i)^* = -\partial_i \begin{cases} q^{\pm 2(N-i+1)} & \text{if } H = U_q sl(N) \\ q^{\pm N + \rho_i} & \text{if } H = U_q so(N) \end{cases} \quad (4.13)$$

if $|q| = 1$, and

$$(x^h)^* = x^k g_{kh}, \quad (\partial_i)^* = -\frac{\Lambda^{\pm 2}}{q^{\pm N} + q^{\pm 2}} \left[(g^{jh} \partial_h \partial_j), x^i \right] \quad (4.14)$$

if $H = U_q so(N)$ and $q \in \mathbb{R}^+$. The upper or lower sign respectively refer to the choices $\epsilon = 1, -1$ in (4.1)₃, and $\Lambda^{\pm 2}$ are respectively defined by

$$\Lambda^{\pm 2} := \left[1 + (q^{\pm 2} - 1) x^i \partial_i + \frac{(q^{\pm 2} - 1)^2}{\omega_n^2} r^2 (g^{ji} \partial_i \partial_j) \right]^{-1}. \quad (4.15)$$

The map φ is a $*$ -homomorphism both for q real and $|q| = 1$. If we denote by φ^\pm its restrictions to $\mathcal{A} \bowtie H^\pm$, then they are $*$ -homomorphisms when $|q| = 1$ (see appendix A.4), and fulfill the relation

$$[\varphi^\pm(g)]^* = \varphi^\mp(g^*) \quad (4.16)$$

when $q \in \mathbb{R}^+$ [13]. Since these relations are of the type considered in proposition 2, the claims of the latter for χ^\pm and their consequences hold.

In particular, when $|q| = 1$ one has a well-defined $*$ on the braided tensor product of $\mathcal{A}_1, \dots, \mathcal{A}_M$ mapping each of the independent, commuting subalgebras $\hat{\mathcal{A}}_\alpha^\pm$ into itself.

Finally, the above results have an important corollary. According to Hochschild cohomology arguments developed by Gerstenhaber [18] and applicable to Heisenberg algebras because of the results found by Du Cloux in Ref. [11], any deformed Heisenberg algebra, in particular the braided tensor products of $\mathcal{A}_1, \dots, \mathcal{A}_M$ considered in this section, can be realized simply by a change of generators in the h -adic completion, $h = \log q$, of its undeformed counterpart (but in general *not* in other, e.g. *operator-norm*, completions). However explicit realizations are not provided by these results. The results presented here, combined to some older ones, allow to determine one such realization. In Ref. [26] Ogievetsky found an explicit realization ϕ or ‘deforming map’ of the elements of $\mathcal{D}_{\epsilon, \mathbf{g}}$ in terms of formal power series in $h = \log q$ with coefficients in the corresponding undeformed Heisenberg algebra. Another, less explicit, one was found in Ref. [15]. The composition of the unbraiding map found in this section, which allows to ‘decouple’ M different copies of $\mathcal{D}_{\epsilon, \mathbf{g}}$ from each other, with the map ϕ provides an explicit realization or ‘deforming map’ of the larger Heisenberg algebra \mathcal{A} (what we have called the ‘braided chain of Heisenberg algebras’), in the h -adic completion of the undeformed $(N \cdot M)$ -dimensional Heisenberg algebra.

5 Formulae for the left action

For psychological reasons we often prefer to work with a left action rather than with a right one. In this section we give the analogs for left H -module algebras of the main results found so far for right H -module algebras. The left action of $g \in H$ on a product fulfills

$$(gg') \triangleright a = g \triangleright (g' \triangleright a), \quad (5.1)$$

$$g \triangleright (aa') = (g_{(1)} \triangleright a)(g_{(2)} \triangleright a'). \quad (5.2)$$

The product laws in the braided tensor product algebras $\hat{\mathcal{A}}^+, \hat{\mathcal{A}}^-$ are respectively given by

$$a_2 a_1 = (\mathcal{R}^{-1(1)} \triangleright a_1) (\mathcal{R}^{-1(2)} \triangleright a_2). \quad (5.3)$$

$$a_2 a_1 = (\mathcal{R}^{(2)} \triangleright a_1) (\mathcal{R}^{(1)} \triangleright a_2), \quad (5.4)$$

The analog of Theorem 1 reads

Theorem 2 *Let $\{H, \mathcal{R}\}$ be a quasitriangular Hopf algebra and H^+, H^- be Hopf subalgebras of H such that $\mathcal{R} \in H^+ \otimes H^-$. Let $\hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2$ be respectively a (left) H^+ - and a H^- -module algebra, so that we can define $\hat{\mathcal{A}}^+$ as in*

(5.3), and $\hat{\varphi}_1^+ : H^+ \ltimes \hat{\mathcal{A}}_1 \rightarrow \hat{\mathcal{A}}_1$ be an algebra homomorphism fulfilling (1.12), so that we can define a map $\hat{\chi}^+ : \hat{\mathcal{A}}_2 \rightarrow \hat{\mathcal{A}}^+$ by

$$\hat{\chi}^+(a_2) := (\mathcal{R}^{(2)} \triangleright a_2) \hat{\varphi}_1^+(\mathcal{R}^{(1)}). \quad (5.5)$$

Alternatively, let $\hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2$ be respectively a (left) H^- - and a H^+ -module algebra, so that we can define $\hat{\mathcal{A}}^-$ as in (5.4), and $\hat{\varphi}_1^+ : H^+ \ltimes \hat{\mathcal{A}}_1 \rightarrow \hat{\mathcal{A}}_1$ be an algebra homomorphism fulfilling (1.12), so that we can define a map $\hat{\chi}^- : \hat{\mathcal{A}}_2 \rightarrow \hat{\mathcal{A}}^-$ by

$$\hat{\chi}^-(a_2) := (\mathcal{R}^{-1(1)} \triangleright a_2) \hat{\varphi}_1^-(\mathcal{R}^{-1(2)}). \quad (5.6)$$

In either case $\hat{\chi}^\pm$ are then injective algebra homomorphisms and

$$[\hat{\chi}^\pm(a_2), \hat{\mathcal{A}}_1] = 0, \quad (5.7)$$

namely the subalgebras $\tilde{\hat{\mathcal{A}}}_2^\pm := \hat{\chi}^\pm(\hat{\mathcal{A}}_2) \approx \hat{\mathcal{A}}_2$ commute with $\hat{\mathcal{A}}_1$. Moreover $\hat{\mathcal{A}}^\pm = \hat{\mathcal{A}}_1 \otimes \tilde{\hat{\mathcal{A}}}_2^\pm$.

The results of section 2 apply without modifications (one just has to place a $\hat{}$ in the appropriate places).

To enumerate the generators of the algebras considered in Sections 3,4 we shall exchange lower with upper indices, so the generators will read $x_{\alpha,i}, \partial^{\alpha,i}$. This is necessary if we wish the x 's to carry what we shall consider the fundamental (vector) representation ρ of $U_q \mathfrak{g}$,

$$g \triangleright x_i = x_j \rho_i^j(g), \quad (5.8)$$

rather than its contragredient $\rho^T \circ S$, because this follows from the row \times column multiplication law $\rho_h^i(gg') = \rho_j^i(g) \rho_h^j(g')$. Apart from this replacement, all the commutation relations remain the same, but can be rephrased in an equivalent way exchanging lower with upper indices also in the braid matrices and in the projectors \mathcal{P}_a , because $\hat{R}^T = \hat{R}$, $\mathcal{P}_a^T = \mathcal{P}_a$. For instance, the analog of (3.1) will read

$$\mathcal{P}_{aij}^{hk} x_h x_k = 0. \quad (5.9)$$

The analogs of (3.2), (4.2) read

$$g \triangleright x_i = \rho_i^j(g) x_j \quad (5.10)$$

$$g \triangleright \partial^i = \partial^h \rho_h^i(Sg). \quad (5.11)$$

Algebra homomorphisms $\hat{\varphi}_1^\pm$ for the algebras considered in Sections 3,4 are immediately obtained in terms of the φ_1^\pm described there, according to the rule

$$\hat{\varphi}_1^\pm(\mathcal{L}^{\pm h}_j) = U^{-1j}_a \varphi_1^\mp(\mathcal{L}^{\mp a}_b) U_h^b. \quad (5.12)$$

Here

$$U_c^b := \rho_c^b(u), \quad (5.13)$$

$u \in H$ is a special element as in (A.1.6), and at the rhs the correct expression in the new notation has lower and upper indices exchanged. If $\hat{\mathcal{A}}_1$ is the quantum Euclidean space \mathbb{R}_q^N one finds, for instance,

$$\hat{\varphi}_1^-(\mathcal{L}_j^{-h}) = U^{-1j}_a g_{ac}[\bar{\mu}^c, x_k]_{q^{-1}} g^{kb} U_b^h \stackrel{(A.1.30)}{=} g_{cj}[\bar{\mu}^c, x_k]_q g^{hk}, \quad (5.14)$$

where μ^c is the same as μ_c [see A.3.3)], but in the new notation. For instance, when $|c| > 1$ it reads

$$\bar{\mu}^c = \bar{\gamma}_c r_{|c|}^{-1} r_{|c|-1}^{-1} x_{-c}, \quad (5.15)$$

with γ_c defined as in (A.3.7) and r_a ($a \geq 0$) defined by the condition

$$r_a^2 = \sum_{h=-a}^a x_h x^h = \sum_{h=-a}^a g^{hk} x_h x_k.$$

The analog of (3.7) is therefore (with $\alpha > 1$)

$$y_{1,i} := x_{1,i} \quad (5.16)$$

$$y^{\alpha,i} := \hat{\chi}^-(x_{\alpha,i}) = x_{\alpha,j} \hat{\varphi}_1(\mathcal{L}_i^{-j}) = x_{\alpha,j} g_{hi}[\bar{\mu}^{1,h}, x_{1,k}]_{q^{-1}} g^{jk}. \quad (5.17)$$

6 Unbraiding ‘chains’ of fuzzy quantum spheres

As a last example, we consider the braided tensor product of M copies $\mathcal{A}_1, \dots, \mathcal{A}_M$ of the q -deformed fuzzy sphere $\hat{S}_{q,N}^2$ [19][‡], which we consider as a left $U_q so(3)$ module algebra. It is generated by x_i fulfilling the relations

$$\begin{aligned} \varepsilon_k^{ij} x_i x_j &= \Lambda_N x_k, \\ g^{ij} x_i x_j &= R^2. \end{aligned} \quad (6.1)$$

Here $R > 0$,

$$C_N = \frac{[N]_q [N+2]_q}{[2]_q^2}, \quad \Lambda_N = R \frac{[2]_q^{N+1}}{\sqrt{[N]_q [N+2]_q}} \quad (6.2)$$

where $[n]_q := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$, and

$$\begin{aligned} \varepsilon_1^{10} &= q^{1/2}, & \varepsilon_1^{01} &= -q^{-1/2}, \\ \varepsilon_0^{00} &= q^{1/2} - q^{-1/2}, & \varepsilon_0^{1-1} &= 1 = -\varepsilon_0^{-11}, \\ \varepsilon_{-1}^{0-1} &= q^{1/2}, & \varepsilon_{-1}^{-10} &= -q^{-1/2} \end{aligned} \quad (6.3)$$

are the spin 1 Clebsch–Gordan coefficients. The multiplet (x_i) carries the fundamental vector representation ρ of $H = U_q so(3)$:

$$g \triangleright x_i = x_j \rho_i^j(g). \quad (6.4)$$

[‡]To relate this to our conventions, the q in [19] should be replaced by $q^{-1/2}$

There is no obvious generalization to higher dimensions, but this algebra appears to be relevant e.g. to D -branes on the $SU(2)$ WZW model [1]. It has a unique irreducible representation, which is equivalent to $Mat(N+1)$. Here we only consider the case $q \in \mathbb{R}^+$, where the star structure is given by $x_i^* = g^{ij} x_j$. Then $\hat{S}_{q,N}^2$ is simply the “discrete series” of Podles’s spheres [27]. It was shown in [19] that there is a star-algebra homomorphism $\hat{\varphi} : H \ltimes \hat{S}_{q,N}^2 \rightarrow \hat{S}_{q,N}^2$, which takes a particularly simple form

$$\begin{aligned}\hat{\varphi}(E^+) &= \frac{1}{R} \sqrt{q^{-1}[2]_q C_N} x_1, \quad \hat{\varphi}(E^-) = -\frac{1}{R} \sqrt{q[2]_q C_N} x_{-1}, \\ \hat{\varphi}(q^{H/2}) &= \frac{[2]_{q^{N+1}}}{[2]_q} \left(1 - \frac{q^{1/2} - q^{-1/2}}{\Lambda_N} x_0 \right)\end{aligned}\tag{6.5}$$

where $E^\pm = X^\pm q^{H/4} \in U_q so(3)$. Note that $(1 - \frac{q^{1/2} - q^{-1/2}}{\Lambda_N} x_0)$ is invertible since the eigenvalues of $q^{H/2}$ are positive (assuming $q > 0$), therefore $\hat{\varphi}(q^{-H/2}) \in \hat{S}_{q,N}^2$ is well-defined also. Hence the algebra homomorphisms $\hat{\varphi}$ is defined on the entire algebra $U_q so(3)$. Using the definition (A.1.15) and the explicit form for the universal \mathcal{R} (see e.g. [7]), one finds

$$[\mathcal{L}^{-i}_j] = \begin{bmatrix} q^{-H/2}, & 0, & 0 \\ -(1-q^{-1})\sqrt{[2]_q} E^-, & 1, & 0 \\ q^{-1/2}(1-q^{-1})^2 q^{-H/2} (E^-)^2, & -(1-q^{-1})\sqrt{[2]_q} q^{-H/2} E^-, & q^{H/2} \end{bmatrix}\tag{6.6}$$

and

$$[\mathcal{L}^{+i}_j] = \begin{bmatrix} q^{-H/2}, & (q-1)\sqrt{[2]_q} q^{-H/2} E^+, & (q-1)^2 q^{-H/2} (E^+)^2, \\ 0, & 1, & q^{-1/2}(q-1)\sqrt{[2]_q} E^+ \\ 0, & 0, & q^{H/2} \end{bmatrix}.\tag{6.7}$$

The unbraiding procedure then works as in Theorem 2. To be specific, assume that the braided tensor product algebra is as in (5.3). Then we set

$$y_{1,i} := x_{1,i}\tag{6.8}$$

$$y_{\alpha,i} := \hat{\chi}(x_{\alpha,i}) = x_{\alpha,j} \hat{\varphi}_1(\mathcal{L}^{+j}_i), \quad \alpha > 1,\tag{6.9}$$

without spelling out these expressions further. According to Theorem 2, they satisfy

Corollary 3

$$\begin{aligned}\varepsilon_k^{ij} y_{\alpha,i} y_{\alpha,j} &= \Lambda_N y_{\alpha,k}, \\ g^{ij} y_{\alpha,i} y_{\alpha,j} &= R^2\end{aligned}$$

for all $\alpha = 1, \dots, M$, together with

$$[y_{1,i}, y_{\alpha,j}] = 0\tag{6.10}$$

$$y_{\alpha,i} y_{\beta,j} = \hat{R}_{ij}^{hk} y_{\beta,h} y_{\alpha,k}\tag{6.11}$$

when $1 < \alpha$ and $\alpha\beta$.

Iterating this procedure as before, we find

Proposition 5 *Let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_M$ be M copies of the $U_q \mathfrak{so}(3)$ -covariant fuzzy quantum sphere. Then $\mathcal{A}_1 \underline{\otimes}^\pm \mathcal{A}_2 \underline{\otimes}^\pm \dots \underline{\otimes}^\pm \mathcal{A}_M = \mathcal{A}_1 \otimes \tilde{\mathcal{A}}_2^\pm \otimes \dots \otimes \tilde{\mathcal{A}}_M^\pm$, where $\tilde{\mathcal{A}}_2^\pm, \dots, \tilde{\mathcal{A}}_M^\pm$ are subalgebras of the lhs isomorphic to \mathcal{A}_1 .*

A Appendix

A.1 The universal R -matrix

In this appendix we recall the basics about the universal R -matrix [9] of the quantum groups $U_q \mathfrak{g}$, while fixing our conventions. Recall that the universal R -matrix \mathcal{R} is a special element

$$\mathcal{R} \equiv \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \in U_q \mathfrak{g} \otimes U_q \mathfrak{g} \quad (\text{A.1.1})$$

intertwining between Δ and opposite coproduct Δ^{op} , and so does also \mathcal{R}_{21}^{-1} :

$$\begin{aligned} \mathcal{R}(g_{(1)} \otimes g_{(2)}) &= (g_{(2)} \otimes g_{(1)})\mathcal{R}, \\ \mathcal{R}_{21}^{-1}(g_{(1)} \otimes g_{(2)}) &= (g_{(2)} \otimes g_{(1)})\mathcal{R}_{21}^{-1}. \end{aligned} \quad (\text{A.1.2})$$

In (A.1.1) we have used a Sweedler notation with upper indices: the right-hand side is a short-hand notation for a sum $\sum_I \mathcal{R}_I^{(1)} \otimes \mathcal{R}_I^{(2)}$ of infinitely many terms. We recall some useful formulae

$$(\Delta \otimes \text{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\text{A.1.3})$$

$$(\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \quad (\text{A.1.4})$$

$$(S \otimes \text{id})\mathcal{R} = \mathcal{R}^{-1} = (\text{id} \otimes S^{-1})\mathcal{R}, \quad (\text{A.1.5})$$

$$S^{-1}(g) = u^{-1}S(g)u. \quad (\text{A.1.6})$$

Here u is any of the elements u_1, u_2, \dots, u_8 defined below:

$$\begin{aligned} u_1 &:= (S\mathcal{R}^{(2)})\mathcal{R}^{(1)} & u_2 &:= (S\mathcal{R}^{-1(1)})\mathcal{R}^{-1(2)} \\ u_3 &:= \mathcal{R}^{(2)}S^{-1}\mathcal{R}^{(1)} & u_4 &:= \mathcal{R}^{-1(1)}S^{-1}\mathcal{R}^{-1(2)} \\ (u_5)^{-1} &:= \mathcal{R}^{(1)}S\mathcal{R}^{(2)} & (u_6)^{-1} &:= (S^{-1}\mathcal{R}^{(1)})\mathcal{R}^{(2)} \\ (u_7)^{-1} &:= \mathcal{R}^{-1(2)}S\mathcal{R}^{-1(1)} & (u_8)^{-1} &:= (S^{-1}\mathcal{R}^{-1(2)})\mathcal{R}^{-1(1)} \end{aligned} \quad (\text{A.1.7})$$

In fact, using the results of Drinfel'd [9, 10] one can show that

$$u_1 = u_3 = u_7 = u_8 = vu_2 = vu_4 = vu_5 = vu_6, \quad (\text{A.1.8})$$

where v is a suitable element belonging to the center of $U_q \mathfrak{so}(N)$.

From (A.1.2) and (A.1.3, A.1.4) it follows the universal Yang-Baxter relation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \quad (\text{A.1.9})$$

whence the other two relations follow

$$\mathcal{R}^{-1}_{12}\mathcal{R}^{-1}_{13}\mathcal{R}^{-1}_{23} = \mathcal{R}^{-1}_{23}\mathcal{R}^{-1}_{13}\mathcal{R}^{-1}_{12}, \quad (\text{A.1.10})$$

$$\mathcal{R}_{13}\mathcal{R}_{23}\mathcal{R}^{-1}_{12} = \mathcal{R}^{-1}_{12}\mathcal{R}_{23}\mathcal{R}_{13}. \quad (\text{A.1.11})$$

As before, let ρ be the fundamental N -dimensional representation of $\mathfrak{g} = sl(N), so(N), sp(N)$. By applying $\text{id} \otimes \rho_c^a \otimes \rho_d^b$ to (A.1.9), $\rho_c^a \otimes \rho_d^b \otimes \text{id}$ to (A.1.10) and $\rho_c^a \otimes \text{id} \otimes \rho_d^b$ to (A.1.11) we respectively find the commutation relations

$$\hat{R}_{cd}^{ab} \mathcal{L}_f^{+d} \mathcal{L}_e^{+c} = \mathcal{L}_c^{+b} \mathcal{L}_d^{+a} \hat{R}_{ef}^{dc}, \quad (\text{A.1.12})$$

$$\hat{R}_{cd}^{ab} \mathcal{L}_f^{-d} \mathcal{L}_e^{-c} = \mathcal{L}_c^{-b} \mathcal{L}_d^{-a} \hat{R}_{ef}^{dc}, \quad (\text{A.1.13})$$

$$\hat{R}_{cd}^{ab} \mathcal{L}_f^{+d} \mathcal{L}_e^{-c} = \mathcal{L}_c^{-b} \mathcal{L}_d^{+a} \hat{R}_{ef}^{dc}, \quad (\text{A.1.14})$$

where $\mathcal{L}_l^{\pm a}$ are the Faddeev-Reshetikin-Takhtadjan generators [12] of $U_q \mathfrak{g}$, defined by

$$\mathcal{L}_l^{+a} := \mathcal{R}^{(1)} \rho_l^a(\mathcal{R}^{(2)}) \quad \mathcal{L}_l^{-a} := \rho_l^a(\mathcal{R}^{-1(1)}) \mathcal{R}^{-1(2)}. \quad (\text{A.1.15})$$

It is known [12] that $\{\mathcal{L}_j^{+i}, \mathcal{L}_j^{-i}\}$ and the square roots of the elements $\mathcal{L}_i^{\pm i}$ provide a (overcomplete) set of generators of $U_q \mathfrak{g}$. Since in our conventions

$$\mathcal{R} \in H^+ \otimes H^-, \quad (\text{A.1.16})$$

then $\mathcal{L}_l^{+a} \in H^+$ and $\mathcal{L}_l^{-a} \in H^-$. Beside (A.1.12-A.1.14) these generators fulfill

$$\mathcal{L}_j^{+i} = 0, \quad \text{if } i > j \quad (\text{A.1.17})$$

$$\mathcal{L}_j^{-i} = 0, \quad \text{if } i < j \quad (\text{A.1.18})$$

$$\mathcal{L}_i^{-i} \mathcal{L}_i^{+i} = \mathcal{L}_i^{+i} \mathcal{L}_i^{-i} = 1, \quad \forall i \quad (\text{A.1.19})$$

$$\mathcal{L}_{-n}^{\pm n} \dots \mathcal{L}_n^{\pm n} = 1, \quad (\text{A.1.20})$$

and, when $\mathfrak{g} = so(N), sp(N)$, some additional relations. When $\mathfrak{g} = so(N)$ the latter read

$$\mathcal{L}_j^{\pm i} \mathcal{L}_k^{\pm h} g^{kj} = g^{hi} \quad \mathcal{L}_i^{\pm j} \mathcal{L}_h^{\pm k} g_{kj} = g_{hi}, \quad (\text{A.1.21})$$

where g_{ij} has been defined in (3.5). The braid matrix \hat{R} is related to \mathcal{R} by $\hat{R}_{hk}^{ij} \equiv R_{hk}^{ji} := (\rho_h^j \otimes \rho_k^i) \mathcal{R}$. With the indices' convention described in sections 3, 4 \hat{R} is given by

$$\hat{R} = q^{-\frac{1}{N}} \left[q \sum_i e_i^i \otimes e_i^i + \sum_{i \neq j} e_i^j \otimes e_j^i + k \sum_{i < j} e_i^i \otimes e_j^j \right] \quad (\text{A.1.22})$$

when $\mathfrak{g} = sl(N)$, and by

$$\begin{aligned} \hat{R} = & q \sum_{i \neq 0} e_i^i \otimes e_i^i + \sum_{\substack{i \neq j, -j \\ \text{or } i=j=0}} e_i^j \otimes e_j^i + q^{-1} \sum_{i \neq 0} e_i^{-i} \otimes e_{-i}^i \quad (\text{A.1.23}) \\ & + k \left(\sum_{i < j} e_i^i \otimes e_j^j - \sum_{i < j} q^{-\rho_i + \rho_j} e_i^{-j} \otimes e_{-i}^j \right) \end{aligned}$$

when $\mathfrak{g} = so(N)$. Here e_j^i is the $N \times N$ matrix with all elements equal to zero except for a 1 in the i th column and j th row. The braid matrix of $sl(N)$ admits the orthogonal projector decomposition

$$\hat{R} = q\mathcal{P}_S - q^{-1}\mathcal{P}_a, \quad \mathfrak{g} = sl(N); \quad (\text{A.1.24})$$

$\mathcal{P}_a, \mathcal{P}_S$ are the $U_q sl(N)$ -covariant deformed antisymmetric and symmetric projectors. The braid matrix of $so(N)$ admits the orthogonal projector decomposition

$$\hat{R} = q\mathcal{P}_s - q^{-1}\mathcal{P}_a + q^{1-N}\mathcal{P}_t \quad \mathfrak{g} = so(N); \quad (\text{A.1.25})$$

$\mathcal{P}_a, \mathcal{P}_t, \mathcal{P}_s$ are the q -deformed antisymmetric, trace, trace-free symmetric projectors.

The compact section of $U_q \mathfrak{g}$ requires $q \in \mathbb{R}^+$ if $\mathfrak{g} = so(N)$, $q \in \mathbb{R}$ if $\mathfrak{g} = sl(N)$ and is characterized by the $*$ -structure

$$(\mathcal{L}^{\pm i}_j)^* = S \mathcal{L}^{\mp j}_i. \quad (\text{A.1.26})$$

For $\mathfrak{g} = so(N)$ this amounts to

$$(\mathcal{L}^{\pm i}_j)^* = g_{ih} \mathcal{L}^{\mp h}_k g^{kj}. \quad (\text{A.1.27})$$

The non-compact sections of $U_q \mathfrak{g}$ require $|q| = 1$ and are characterized by the $*$ -structure

$$(\mathcal{L}^{\pm i}_j)^* = U^{-1i}_r \mathcal{L}^{\pm r}_s U^s_j = u \mathcal{L}^{\pm i}_j u^{-1}. \quad (\text{A.1.28})$$

This can be checked using the property $(\hat{R}^{ij}_{hk})^* = \hat{R}^{-1ji}_{kh}$. Here we have defined

$$U^i_j = \rho^i_j(u) \quad (\text{A.1.29})$$

with u any of the elements defined in (A.1.7). For $\mathfrak{g} = so(N)$ one can take

$$U^i_j := g^{ih} g_{jh}. \quad (\text{A.1.30})$$

From formulae (A.1.3), (A.1.4) in the Appendix A.1 one finds that the coproducts are given by

$$\Delta(\mathcal{L}^{+i}_j) = \mathcal{L}^{+i}_h \otimes \mathcal{L}^{+h}_j \quad \Delta(\mathcal{L}^{-i}_j) = \mathcal{L}^{-i}_h \otimes \mathcal{L}^{-h}_j. \quad (\text{A.1.31})$$

A.2 Proof of Proposition 1

We make use of the identity

$$\varphi^\pm(g^\pm) \triangleleft h^\pm = \varphi^\pm(g^\pm \triangleleft h^\pm), \quad (\text{A.2.1})$$

for any $g^\pm, h^\pm \in H^\pm$, which we prove in Ref. [16]. The right action appearing at the rhs is the (right) adjoint action on itself

$$h \triangleleft g = Sg_{(1)}hg_{(2)}, \quad g, h \in H; \quad (\text{A.2.2})$$

where S denotes the antipode of the Hopf algebra H . We shall also need the inverse of (1.9),

$$a_1a_2 = (a_2 \triangleleft \mathcal{R}^{-1(2)}) (a_1 \triangleleft \mathcal{R}^{-1(1)}). \quad (\text{A.2.3})$$

Now,

$$\begin{aligned} \chi^+(a_2) &\stackrel{(1.13)}{=} \varphi_1^+(\mathcal{R}^{(1)}) (a_2 \triangleleft \mathcal{R}^{(2)}) \\ &\stackrel{(A.2.3)}{=} (a_2 \triangleleft \mathcal{R}^{(2)} \mathcal{R}^{-1(2')}) [\varphi_1^+(\mathcal{R}^{(1)}) \triangleleft \mathcal{R}^{-1(1')}] \\ &\stackrel{(A.2.1)}{=} (a_2 \triangleleft \mathcal{R}^{(2)} \mathcal{R}^{-1(2')}) \varphi_1^+(\mathcal{R}^{(1)} \triangleleft \mathcal{R}^{-1(1')}) \\ &\stackrel{(A.2.2)}{=} (a_2 \triangleleft \mathcal{R}^{(2)} \mathcal{R}^{-1(2')}) \varphi_1^+(S\mathcal{R}_{(1)}^{-1(1')} \mathcal{R}^{(1)} \mathcal{R}_{(2)}^{-1(1')}) \\ &\stackrel{(A.1.3)}{=} (a_2 \triangleleft \mathcal{R}^{(2)} \mathcal{R}^{-1(2')} \mathcal{R}^{-1(2'')}) \varphi_1^+(S\mathcal{R}^{-1(1'')} \mathcal{R}^{(1)} \mathcal{R}^{-1(1')}) \\ &= (a_2 \triangleleft \mathcal{R}^{-1(2'')}) \varphi_1^+(S\mathcal{R}^{-1(1'')}), \end{aligned}$$

which proves (1.21). Similarly one proves (1.22).

A.3 The maps φ^\pm for the quantum Euclidean spaces or spheres

We introduce the short-hand notation

$$[A, B]_x = AB - xBA. \quad (\text{A.3.1})$$

In Ref. [4] we have found algebra homomorphisms $\varphi^\pm : \mathbb{R}_q^N \rtimes U_q^\pm so(N) \rightarrow \mathbb{R}_q^N$. The images of φ^- on the negative FRT generators read

$$\varphi^-(\mathcal{L}^{-i}_j) = g^{ih} [\mu_h, x^k]_q g_{kj}, \quad (\text{A.3.2})$$

where

$$\begin{aligned} \mu_0 &= \gamma_0(x^0)^{-1} && \text{for } N \text{ odd,} \\ \mu_{\pm 1} &= \gamma_{\pm 1}(x^{\pm 1})^{-1} \mathcal{L}^{\pm 1}_1 && \text{for } N \text{ even,} \\ \mu_a &= \gamma_a r_{|a|}^{-1} r_{|a|-1}^{-1} x^{-a} && \text{otherwise,} \end{aligned} \quad (\text{A.3.3})$$

and $\gamma_a \in \mathbb{C}$ are normalization constants fulfilling the conditions

$$\begin{aligned}\gamma_0 &= -q^{-\frac{1}{2}}h^{-1} && \text{for } N \text{ odd,} \\ \gamma_1\gamma_{-1} &= \begin{cases} -q^{-1}h^{-2} & \text{for } N \text{ odd,} \\ k^{-2} & \text{for } N \text{ even,} \end{cases} && (A.3.4) \\ \gamma_a\gamma_{-a} &= -q^{-1}k^{-2}\omega_a\omega_{a-1} && \text{for } a > 1.\end{aligned}$$

h, k, ω_a are defined as in Sections 3, 4. On the other hand, the images of φ^+ on the positive FRT generators read

$$\varphi^+(\mathcal{L}^{+i}_j) = g^{ih}[\bar{\mu}_h, x^k]_{q^{-1}}g_{kj}, \quad (A.3.5)$$

where

$$\begin{aligned}\bar{\mu}_0 &= \bar{\gamma}_0(x^0)^{-1} && \text{for } N \text{ odd,} \\ \bar{\mu}_{\pm 1} &= \bar{\gamma}_{\pm 1}(x^{\pm 1})^{-1}\mathcal{L}^{\mp 1}_1 && \text{for } N \text{ even,} \\ \bar{\mu}_a &= \bar{\gamma}_a r_{|a|}^{-1} r_{|a|-1}^{-1} x^{-a} && \text{otherwise,}\end{aligned} \quad (A.3.6)$$

and $\bar{\gamma}_a \in \mathbb{C}$ normalization constants fulfilling the conditions

$$\begin{aligned}\bar{\gamma}_0 &= q^{\frac{1}{2}}h^{-1} && \text{for } N \text{ odd,} \\ \bar{\gamma}_1\bar{\gamma}_{-1} &= \begin{cases} -qh^{-2} & \text{for } N \text{ odd,} \\ k^{-2} & \text{for } N \text{ even,} \end{cases} && (A.3.7) \\ \bar{\gamma}_a\bar{\gamma}_{-a} &= -qk^{-2}\omega_a\omega_{a-1} && \text{for } a > 1.\end{aligned}$$

Incidentally, for odd N one can choose the free parameters $\gamma_a, \bar{\gamma}_a$ in such a way that φ^+, φ^- can be ‘glued’ into an algebra homomorphism $\varphi : \mathbb{R}_q^N \rtimes U_{q\text{so}}(N) \rightarrow \mathbb{R}_q^N$ [4].

We give the explicit expression for $\varphi^\pm(\mathcal{L}^{\pm i}_j)$ in the case $N = 3$:

$$[\varphi^-(\mathcal{L}^{-i}_j)] = \begin{bmatrix} -qh\gamma_1(x^0)^{-1}r & & \\ q^{\frac{1}{2}}(q+1)(x^0)^{-1}x^+ & 1 & \\ q^{\frac{1}{2}}(q+1)(h\gamma_1rx^0)^{-1}(x^+)^2 & (1+q^{-1})(h\gamma_1r)^{-1} & -(qh\gamma_1r)^{-1}x^0 \end{bmatrix} \quad (A.3.8)$$

and

$$[\varphi^+(\mathcal{L}^{+i}_j)] = \begin{bmatrix} -h\bar{\gamma}_1r^{-1}x^0 & q^{-\frac{1}{2}}\bar{\gamma}_1kr^{-1}x^- & q^{-2}k\bar{\gamma}_1(rx^0)^{-1}(x^-)^2 \\ & 1 & q^{-\frac{1}{2}}(q^{-1}+1)(x^0)^{-1}x^- \\ & & -(h\bar{\gamma}_1x^0)^{-1}r \end{bmatrix} \quad (A.3.9)$$

When $q \in \mathbb{R}^+$ the real structure of \mathbb{R}_q^N is given by

$$(x^i)^* = x^j g_{ji}. \quad (A.3.10)$$

Note that when N is odd $\mu_0, \bar{\mu}_0$, which are completely determined by their definitions, are such that $\mu_0^* = -q^{-1}\bar{\mu}_0$. We fix the other $\gamma_a, \bar{\gamma}_a$ so that for any a

$$\mu_a^* = -q^{-1}g_{ab}\bar{\mu}_b. \quad (\text{A.3.11})$$

This was already considered in Ref. [4] and requires

$$\begin{aligned} \gamma_{\pm 1}^* &= -\bar{\gamma}_{\mp 1} && \text{if } N \text{ even} \\ \gamma_a^* &= -\bar{\gamma}_{-a} \begin{cases} 1 & \text{if } a < 0 \\ q^{-2} & \text{if } a > 0 \end{cases} && \text{otherwise.} \end{aligned} \quad (\text{A.3.12})$$

As a consequence,

$$\begin{aligned} [\varphi^-(\mathcal{L}_j^{-i})]^* &\stackrel{(\text{A.3.2})}{=} \left(g^{ih}[\mu_h, x^k]_q g_{kj} \right)^* \\ &\stackrel{(\text{A.3.10})}{=} g^{ih} [x^j, \mu_h^*]_q \\ &\stackrel{(\text{A.3.11})}{=} [\bar{\mu}_i, x^j]_{q^{-1}} \\ &\stackrel{(\text{A.3.5})}{=} g_{ih} \varphi^+(\mathcal{L}_k^{+h}) g^{kj} \\ &\stackrel{(\text{A.1.27})}{=} \varphi^+[(\mathcal{L}_j^{-i})^*] \end{aligned}$$

In other words

$$[\varphi^\pm(g)]^* = \varphi^\mp(g^*). \quad (\text{A.3.13})$$

When $|q| = 1$

$$(x^i)^* = x^i \quad (\text{A.3.14})$$

Note that when N is odd $\mu_0, \bar{\mu}_0$, which are completely determined by their definitions, are such that $\mu_0^* = -q\mu_0 = \bar{\mu}_0$. We fix the other $\gamma_a, \bar{\gamma}_a$ so that for any a

$$\mu_a^* = -q\mu_a, \quad \bar{\mu}_a^* = -q^{-1}\bar{\mu}_a. \quad (\text{A.3.15})$$

This requires

$$\begin{aligned} \gamma_{\pm 1}^* &= -\gamma_{\pm 1} && \text{if } N \text{ even} \\ \gamma_a^* &= -\gamma_a \begin{cases} 1 & \text{if } a < 0 \\ q^{-2} & \text{if } a > 0 \end{cases} && \text{otherwise.} \end{aligned} \quad (\text{A.3.16})$$

As a consequence,

$$\begin{aligned} [\varphi^-(\mathcal{L}_j^{-i})]^* &\stackrel{(\text{A.3.2})}{=} \left(g^{ih}[\mu_h, x^k]_q g_{kj} \right)^* \\ &\stackrel{(\text{A.3.14})}{=} -q^{-1} g^{hi} [\mu_h^*, x^k]_q g_{jk} \\ &\stackrel{(\text{A.3.15})}{=} g^{hi} [\mu_h, x^k]_q g_{jk} \\ &\stackrel{(\text{A.1.30}), (\text{A.3.2})}{=} U_r^{-1i} \varphi_r^-(\mathcal{L}_s^{-r}) U_r^i \\ &\stackrel{(\text{A.1.28})}{=} \varphi^-[(\mathcal{L}_j^{-i})^*]. \end{aligned}$$

Similarly one proves that $[\varphi^-(\mathcal{L}_j^{-i})]^* = \varphi^-[(\mathcal{L}_j^{-i})^*]$. In other words, φ^\pm are *-homomorphisms.

A.4 The maps φ for the deformed Heisenberg algebras

In Ref. [13] we constructed an algebra homomorphism $\varphi : U_{q\mathfrak{so}(N)} \bowtie \mathcal{A}_1 \rightarrow \mathcal{A}_1$, where \mathcal{A}_1 denotes the $U_{q\mathfrak{so}(N)}$ -covariant (deformed) Heisenberg algebra, such that φ is a $*$ -homomorphism

$$\varphi(g^*) = \varphi(g)^* \quad (\text{A.4.1})$$

on the compact section of $U_{q\mathfrak{so}(N)}$ (what requires $q \in \mathbb{R}^+$). One can easily prove the same result also for the noncompact section (A.1.28) of $\mathfrak{g} = \mathfrak{so}(N)$ as well as the compact and noncompact sections of $\mathfrak{g} = \mathfrak{sl}(N)$. This can be done maybe most rapidly using as a set of generators the so-called "vector fields" Z_j^i [29], which are related to the FRT generators by

$$Z_j^i = \mathcal{L}^{+i}_h S \mathcal{L}^{-h}_j. \quad (\text{A.4.2})$$

From (A.1.26), (A.1.28) one immediately finds

$$(Z_j^i)^* = Z_i^j \quad \text{if } q \in \mathbb{R}^+ \quad (\text{A.4.3})$$

$$(Z_j^i)^* = U^{-1i}_a (S^{-1} \mathcal{L}^{-h}_b) \mathcal{L}^{+a}_h U_j^b \quad \text{if } |q| = 1; \quad (\text{A.4.4})$$

if $\mathfrak{g} = \mathfrak{so}(N)$ the second relation reduces to

$$(Z_j^i)^* = U^{-1a}_b Z_c^b \hat{R}^{-1ci}_{aj}. \quad (\text{A.4.5})$$

In Ref. [8] the explicit expression of $\varphi(Z_j^i)$ in terms of the x 's and ∂ 's is given both for $g = \mathfrak{sl}(N)$ and $\mathfrak{g} = \mathfrak{so}(N)$, and it is not difficult to show that on these generators (and therefore on all of $U_q \mathfrak{g}$) (A.4.1) is satisfied. In performing the calculations one has to keep in mind that the authors of Ref. [8] work with the left action, rather than with the right, so one has to switch to the conventions described in section 5, but, as explained there, this will not modify the result (A.4.1). As an intermediate step, we give the action of the $*$ -structure on the coordinates and derivatives for the case $\mathfrak{g} = \mathfrak{so}(N)$, in the notation used there:

$$(x_h)^* = g^{hk} x_k, \quad (\partial^i)^* = -q^{-N} \hat{\partial}_i \quad \text{if } q \in \mathbb{R}^+ \quad (\text{A.4.6})$$

$$(x_h)^* = x_h, \quad (\partial^i)^* = -q^N U^{-1i}_j \partial^j, \quad (\hat{\partial}_i)^* = -q^{-N} \partial_i \quad \text{if } |q| = 1 \quad (\text{A.4.7})$$

References

- [1] A. Yu. Alekseev, A. Recknagel, V. Schomerus, "Non-commutative World-volume Geometries: Branes on $SU(2)$ and Fuzzy Spheres", *JHEP* 9909, 023 (1999).
- [2] A. Van Daele and S. Van Keer, *Compositio Mathematica* **91**, 201 (1994). A. Borowiec, W. Marcinek, "On crossed product of algebras", math-ph/0007031, and references therein; "Hopf modules and their duals", math.QA/0007151.

- [3] U. Carow-Watamura, M. Schlieker, S. Watamura, “ $SO_q(N)$ covariant differential calculus on quantum space and quantum deformation of Schroedinger equation”, *Z. Physik C - Particles and Fields* **49** (1991) 439.
- [4] B. L. Cerchiai, G. Fiore, J. Madore, “Geometrical Tools for Quantum Euclidean Spaces”, Dip. Matematica e Applicazioni, Napoli 99-52, LMU-TPW 99-17, MPI-PhT/99-45, math.QA/0002007
- [5] B.L. Cerchiai, J. Madore, S. Schraml, J. Wess, “Structure of the Three-dimensional Quantum Euclidean Space”, LMU-TPW 2000-06, MPI-PhT/2000-08, math.QA/0004011.
- [6] B.L. Cerchiai, J. Wess, “ q -Deformed Minkowski Space based on a q -Lorentz Algebra” *Euro. Phys. J.* **C5** 1998, 553.
- [7] V. Chari and A. Pressley, ”A guide to quantum groups”. Cambridge University press, 1994
- [8] C.-S. Chu, B. Zumino, “Realization of vector fields fro quantum groups as pseudodifferential operators on quantum spaces”, *Proc. XX Int. Conf. on Group Theory Methods in Physics*, Toyonaka (Japan), 1995, and q-alg/9502005.
- [9] V. Drinfeld, “Quantum groups,” in *I.C.M. Proceedings, Berkeley*, p. 798. 1986.
- [10] V. Drinfeld, “Quasi Hopf algebras,” *Lenin. Math. Jour.* **1** (1990) 1419.
- [11] F. du Cloux, “Homologie, Groupes Ext^n Représentations de longueur finie des groupes de Lie”, *Asterisque* (Soc. Math. France) **124-125** (1985), 129.
- [12] L.D. Faddeev, N.Y. Reshetikhin, L. Takhtadjan, “Quantization of Lie groups and Lie algebras”, *Leningrad Math. J.* **1** (1990) 193.
- [13] G. Fiore, “Realization of $U_q(so(N))$ within the Differential Algebra on \mathbf{R}_q^N ”, *Commun. Math. Phys.* **169** (1995), 475-500.
- [14] G. Fiore, “Braided Chains of q -Deformed Heisenberg Algebras”, *J. Phys.* **A 31** (1998), 5289.
- [15] G. Fiore, “Drinfel’d Twist and q -Deforming Maps for Lie Group Covariant Heisenberg Algebras”, *Rev. Math. Phys* **12** (2000), 327.
- [16] G. Fiore, “ On the Decoupling of Inhomogeneous and Homogeneous Parts in Inhomogeneous Quantum Groups”, Preprint 00-31 Dip. Matematica e Applicazioni, Università di Napoli.
- [17] G. Fiore, J. Madore, “The geometry of the quantum Euclidean space” *J. Geom. Phys.* **33** (2000), 257-287.
- [18] M. Gerstenhaber, “On the Deformation of Rings and Algebrae”, *Ann. Math.* **79** (1964), 59.
- [19] H. Grosse, J. Madore, H. Steinacker, “Field Theory on the q -deformed Fuzzy Sphere”, hep-th/0005273.

- [20] A. Joyal, R. Street, *Braided Monoidal Categories*, Mathematics Reports 86008, Macquarie University, 1986.
- [21] T. Hayashi, “q-Analogs of Clifford and Weyl Algebras: Spinor and Oscillator Realizations of Quantum Enveloping algebras” *Commun. Math. Phys.* **127** (1990), 129.
- [22] A.N. Kirillov, N. Reshetikhin, ”q- Weyl group and a Multiplicative Formula for Universal R- Matrices” *Comm. Math. Phys.* **134**, 421 (1990)
- [23] S. Majid, *Int. J. Mod. Phys.* **A5**, 1 (1990); *J. Algebra* **130**, 17 (1990); *Lett. Math. Phys.* **22**, 167 (1991); *J. Algebra* **163**, 191 (1994).
- [24] For a review see for instance: S. Majid, *Foundations of Quantum Groups*, Cambridge Univ. Press (1995); and references therein.
- [25] J. Madore, *An introduction to noncommutative differential geometry and its physical applications*. No. 257 in London Mathematical Society Lecture Note Series. Cambridge University Press, second ed., 1999.
- [26] O. Ogievetsky “Differential operators on quantum spaces for $GL_q(n)$ and $SO_q(n)$ ” *Lett. Math. Phys.* **24** (1992), 245.
- [27] P. Podleś, “Quantum spheres”, *Lett. Math. Phys.* **14** (1987), 193.
- [28] W. Pusz, S. L. Woronowicz, “Twisted Second Quantization”, *Rep. Math. Phys.* **27** (1989), 231.
- [29] P. Schupp, P. Watts, B. Zumino “Bicovariant Quantum Algebras and Quantum Lie Algebras”, *Commun. Math. Phys.* **157** (1993), 305; and references therein.
- [30] H. Steinacker, “Quantum Anti-de Sitter space and sphere at roots of unity”, *Adv. Theor. Math. Phys.* **4**, Nr. 1 (2000); hep-th/9910037.
- [31] J. Wess, B. Zumino, “Covariant differential calculus on the quantum hyperplane”, *Nucl. Phys. (Proc. Suppl.)* **18B** (1990) 302.